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Asymptotics of zeros of basic sine and cosine functions

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Abstract

We give an elementary calculus proof of the asymptotic formulas for the zeros of the q-sine and cosine functions which have been recently found numerically by Gosper and Suslov. Monotone convergent sequences of the lower and upper bounds for these zeros are constructed as an extension of our method. Improved asymptotics are found by a different method using the Lagrange inversion formula. Asymptotic formulas for the points of inflection of the basic sine and cosine functions are conjectured. Analytic continuation of the q-zeta function is discussed as an application. An interpretation of the zeros is given.

Keywords: Basic trigonometric functions; Basic Fourier series; Asymptotics of zeros of q-trigonometric functions; Lagrange inversion formula; q-zeta function

1. Introduction

This paper continues a series of papers dedicated to investigation of the basic Fourier series introduced recently by Bustoz and Suslov [4]. See [4,7,11,22,23,25] for an introduction to the theory of q-Fourier series, [12–15,20,21] regarding the corresponding basic exponential function on a q-quadratic grid [18], a review article [22], and a forthcoming monograph [26]. The case of a q-linear lattice is investigated in [3]. In the current paper we give a rigorous proof of the asymptotic formulas for

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the zeros of the basic sine and cosine functions numerically found in [7]. Improved asymptotics are found by a different method.

The q-sine and cosine functions under consideration can be introduced as

$$S_q(\eta;\omega) = \frac{(-i\omega;q^{1/2})_\infty - (i\omega;q^{1/2})_\infty}{2i(-q\omega^2;q^2)_\infty} = \frac{1}{(-q\omega^2;q^2)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1/2)}}{(q^{1/2};q^{1/2})_{2k+1}} \omega^{2k+1} \tag{1.1}$$

and

$$C_q(\eta;\omega) = \frac{(-i\omega;q^{1/2})_\infty + (i\omega;q^{1/2})_\infty}{2(-q\omega^2;q^2)_\infty} = \frac{1}{(-q\omega^2;q^2)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k-1/2)}}{(q^{1/2};q^{1/2})_{2k}} \omega^{2k}. \tag{1.2}$$

Functions $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ are special cases of the q-sine $S_q(x;\omega)$ and q-cosine $C_q(x;\omega)$ functions in two independent variables x and ω when $x=\eta=(q^{1/4}+q^{-1/4})/2$; see [4,26] for more details. We use the standard notations [5] for the basic hypergeometric series and for the q-shifted factorials throughout the paper.

The ω-zeros of the $S_q(\eta;\omega)$ are the eigenvalues related to the basic Fourier series on a q-quadratic grid [4,26]. Their asymptotics are very important for investigation of the convergence of these series [24]. The main properties of these zeros were discussed in [4,9,10] from different viewpoints. We remind the reader that when $0<q<1$ all zeros of $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ are real. Also these zeros are simple, the positive zeros of the basic sine function $S_q(\eta;\omega)$ are interlaced with those of the basic cosine function $C_q(\eta;\omega)$; see Theorems 1–4 of [4] or Section 5.4 of [26]. Asymptotic behavior of the large zeros of these q-trigonometric functions has been discussed in Theorems 5 and 6 of [4]; see also [7] for numerical investigation of these zeros.

Let $0=\omega_0<\omega_1<\omega_2<\omega_3<\cdots$ be positive zeros of $S_q(\eta;\omega)$ and let $\varpi_1<\varpi_2<\varpi_3<\cdots$ be positive zeros of $C_q(\eta;\omega)$. Gosper and Suslov [7] have found numerically the following asymptotic formulas:

$$\omega_n = q^{1/4-n} - c_1(q) + o(1) \tag{1.3}$$

and

$$\varpi_n = q^{3/4-n} - c_1(q) + o(1) \tag{1.4}$$

as $n\to\infty$. Here,

$$c_1(q) = \frac{q^{1/4}}{2(1-q^{1/2})} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} = \frac{q^{1/4}(1+q^{1/2})(1+q)}{2\Gamma_{q^2}^2(1/2)} \tag{1.5}$$

and $\Gamma_{q^2}(z)$ is the q-gamma function. Function $c_1(q)$ is nonnegative and increasing on $[0,1]$; see [7] or Fig. 6 below for the graph of this function and Appendix C for the proof of the monotonicity. The maximum value of this function on $[0,1]$ is

$$\lim_{q\to 1^-} c_1(q) = 2/\pi \simeq 0.63661977236758. \tag{1.6}$$

Numerical analysis in [7] has shown that asymptotic formulas (1.3)–(1.4) are pretty accurate.

Our main objective in the current paper is to present a rigorous proof of these formulas and to find the next terms in these asymptotic expansions. The paper is organized as follows. In Section 2 we discuss some properties of the basic trigonometric functions $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ and then give elementary calculus proofs of the asymptotic formulas (1.3)–(1.4) and some of their modifications in Section 3. In Section 4 we construct monotone convergent sequences of the lower and upper bounds for all positive zeros of these q-sine and cosine functions as an extension of our method. In Section 5 we discuss the points of inflection of the basic sine and cosine functions. In Section 6 we give a mechanical interpretation of the zeros. Improved asymptotics of the zeros, which are the main result of this paper, are established in Section 7 on the basis of the Lagrange inversion formula. The last section is devoted to an application. Here we apply the above asymptotics of the zeros in order to present an analytic continuation of the q-zeta function $\zeta_q(z)$ originally introduced in [24] for the half-plane $\operatorname{Re} z > 1$ to the larger domain $\operatorname{Re} z > -3$. Several useful asymptotic formulas and estimates needed for the investigation of the q-Fourier series are derived in Appendix A. Alternative forms of a constant in the new asymptotic formulas from Section 7 are derived in Appendix B.

2. Some properties of q-sine and cosine functions

In this section we remind the reader main properties of the basic sine and cosine functions that are important for further consideration. The large ω-asymptotics of the $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ can be investigated on the basis of the following expressions:

$$\begin{aligned} C_q(\eta;\omega) = {} & \frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty}{(q^{1/2};q)_\infty (q, -q\omega^2, -q/\omega^2; q^2)_\infty} C_q\left(\eta; \frac{q^{1/2}}{\omega}\right) \\ & - \omega \frac{(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(q^{1/2};q)_\infty (q, -q\omega^2, -q/\omega^2; q^2)_\infty} S_q\left(\eta; \frac{q^{1/2}}{\omega}\right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} S_q(\eta;\omega) = {} & \omega \frac{(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(q^{1/2};q)_\infty (q, -q\omega^2, -q/\omega^2; q^2)_\infty} C_q\left(\eta; \frac{q^{1/2}}{\omega}\right) \\ & + \frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty}{(q^{1/2};q)_\infty (q, -q\omega^2, -q/\omega^2; q^2)_\infty} S_q\left(\eta; \frac{q^{1/2}}{\omega}\right). \end{aligned} \tag{2.2}$$

These formulas follow directly from (4.3) and (4.4) of [7], see also (5.16) and (5.17) of [4], when we substitute $e^{i\theta} = q^{1/4}$. One can easily see that Eqs. (2.1)–(2.2) and

(1.1)–(1.2) determine the asymptotic behavior of the basic trigonometric functions $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ for the large values of the variable ω.

It has been shown in [7] that the graphs of the $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$ look much more elegant if we choose a different normalization for these functions. An analog of the main trigonometric identity for the basic trigonometric functions established in [4] is

$$C_q^2(\eta;\omega) + S_q^2(\eta;\omega) = \frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}. \tag{2.3}$$

Introducing functions

$$F(\omega) = \sqrt{\frac{(-q\omega^2;q^2)_\infty}{(-\omega^2;q^2)_\infty}}\, S_q(\eta;\omega), \tag{2.4}$$

$$G(\omega) = \sqrt{\frac{(-q\omega^2;q^2)_\infty}{(-\omega^2;q^2)_\infty}}\, C_q(\eta;\omega), \tag{2.5}$$

one can rewrite (2.3) as

$$F^2(\omega) + G^2(\omega) = 1. \tag{2.6}$$

The Wronskian of these functions has a simple explicit form

$$\kappa(\omega) := G(\omega)F'(\omega) - G'(\omega)F(\omega) = \sum_{k=0}^{\infty} \frac{q^{k/2}}{1+\omega^2 q^k} \tag{2.7}$$

and the following differentiation formulas hold:

$$F'(\omega) = \kappa(\omega)G(\omega), \quad G'(\omega) = -\kappa(\omega)F(\omega). \tag{2.8}$$

See [7,26] for more details.

Functions $F(\omega)$ and $G(\omega)$ have the same real zeros as our original functions, the $S_q(\eta;\omega)$ and $C_q(\eta;\omega)$, but they obey nice properties similar to those for the classical trigonometric functions [7]. For example, functions $F(\omega)$ and $G(\omega)$ are bounded for all real values of ω and change from -1 to 1. Moreover, all extrema of function $F(\omega)$ ($G(\omega)$) are located at zeros of $G(\omega)$ ($F(\omega)$); function $F(\omega)$ ($G(\omega)$) is monotone between any two successive zeros of $G(\omega)$ ($F(\omega)$). These properties are direct consequences of (2.6)–(2.8). See [7] and Section 11.1 of [26] for the graphs of $F(\omega)$ and $G(\omega)$ for different values of parameter q. It is worth mentioning that both functions, $F(\omega)$ and $G(\omega)$, satisfy the following differential equation:

$$u'' + \kappa^2 u = (\log\kappa)' u'. \tag{2.9}$$

We shall use these properties of the functions $F(\omega)$ and $G(\omega)$ in order to prove the asymptotic formulas (1.3)–(1.4).

One can easily see from (2.7) that

$$\kappa'(\omega) = -2\omega \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^2}, \tag{2.10}$$

which means that the $\kappa(\omega)$ is increasing for all negative values of the ω, decreasing for all positive ones, and attains its maximum at $\omega = 0$. The large ω-asymptotic of $\kappa(\omega)$ can be found from the expansion [24]

$$\kappa(\omega) = \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \frac{(-q^{1/2}\omega^2, -q^{1/2}/\omega^2;q)_\infty}{(-\omega^2, -q/\omega^2;q)_\infty} - \frac{q^{1/2}}{\omega^2}\kappa\left(\frac{q^{1/2}}{\omega}\right), \tag{2.11}$$

which is an easy consequence of the Ramanujan ${}_1\psi_1$-summation formula; see, for example, [5]. We remind the reader that the $\kappa(\omega)$ determines the $\mathscr{L}^2$-norm of the basic trigonometric system up to a factor [4,24,26].

The large ω-asymptotic of the $\kappa'(\omega)$ follows from

$$\begin{aligned} \kappa'(\omega) = &-\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \frac{(-q^{1/2}\omega^2, -q^{1/2}/\omega^2;q)_\infty}{\omega(-\omega^2, -q/\omega^2;q)_\infty} \\ &- (q;q)_\infty^2 (q^{1/2};q^{1/2})_\infty^2\, \omega \frac{(q^{1/2}\omega^2, 1/\omega^2;q^{1/2})_\infty}{(-\omega^2, -q/\omega^2;q)_\infty^2} \\ &+ \frac{2q^{1/2}}{\omega^3} \sum_{k=0}^{\infty} \frac{q^{k/2}}{(1+q^{1+k}/\omega^2)^2}. \end{aligned} \tag{2.12}$$

We shall derive this formula in Appendix A together with asymptotic expression for the $\kappa''(\omega)$ and uniform bounds for these derivatives.

3. Proof of asymptotic formulas

In this section we give a rigorous proof of (1.3)–(1.4) by means of, essentially, elementary calculus tools only. These formulas have been conjectured in [7]. Let us reformulate the main result in the form of a theorem.

Theorem 1. *Let* $0 = \omega_0 < \omega_1 < \omega_2 < \omega_3 < \cdots$ *be positive zeros of* $S_q(\eta;\omega)$ *and let* $\varpi_1 < \varpi_2 < \varpi_3 < \cdots$ *be positive zeros of* $C_q(\eta;\omega)$ *for* $0<q<1$. *Then,*

$$\omega_n = q^{1/4-n} - c_1(q) + o(1), \tag{3.1}$$

$$\varpi_n = q^{3/4-n} - c_1(q) + o(1) \tag{3.2}$$

as $n \to \infty$, *where*

$$c_1(q) = \frac{q^{1/4}}{2(1-q^{1/2})} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{3.3}$$

Proof. Denote

$$\omega_n^{(0)} = q^{1/4-n}. \tag{3.4}$$

In view of (2.1)–(2.2) and (2.4)–(2.5)

$$\frac{F(\omega_n^{(0)})}{G(\omega_n^{(0)})} = \frac{S_q(\eta;\omega_n^{(0)})}{C_q(\eta;\omega_n^{(0)})} = \frac{S_q(\eta;q^{1/2}/\omega_n^{(0)})}{C_q(\eta;q^{1/2}/\omega_n^{(0)})} > 0 \tag{3.5}$$

for all $q^{1/2}/\omega_n^{(0)} < \varpi_1$ or $q^{1/4+n} < \varpi_1$. The last inequality holds for all sufficiently large values of n for any $0<q<1$. This means that for the sufficiently large n we always have $\omega_m < \omega_n^{(0)} < \varpi_{m+1}$, where m, generally speaking, may be different from n (we shall show later that $m=n$ for sufficiently large n). By the Mean Value Theorem for the interval $[\omega_m, \omega_n^{(0)}]$ one can write (see Fig. 1)

$$F(\omega_n^{(0)}) = F'(c)(\omega_n^{(0)} - \omega_m), \tag{3.6}$$

where $c \in (\omega_m, \omega_n^{(0)})$. Hence,

$$\omega_n^{(0)} - \omega_m = \frac{F(\omega_n^{(0)})}{F'(c)}, \tag{3.7}$$

where $F'(c) = \kappa(c)G(c)$ by (2.8).

The following main inequalities hold:

$$0 < \frac{|F(\omega_n^{(0)})|}{\kappa(\omega_m)} < \frac{F(\omega_n^{(0)})}{F'(c)} < \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})G(\omega_n^{(0)})} \tag{3.8}$$

in view of the monotonicity properties

$$\kappa(\omega_m) > \kappa(c) > \kappa(\omega_n^{(0)}), \quad |G(\omega_m)| = 1 > |G(c)| > |G(\omega_n^{(0)})| \tag{3.9}$$

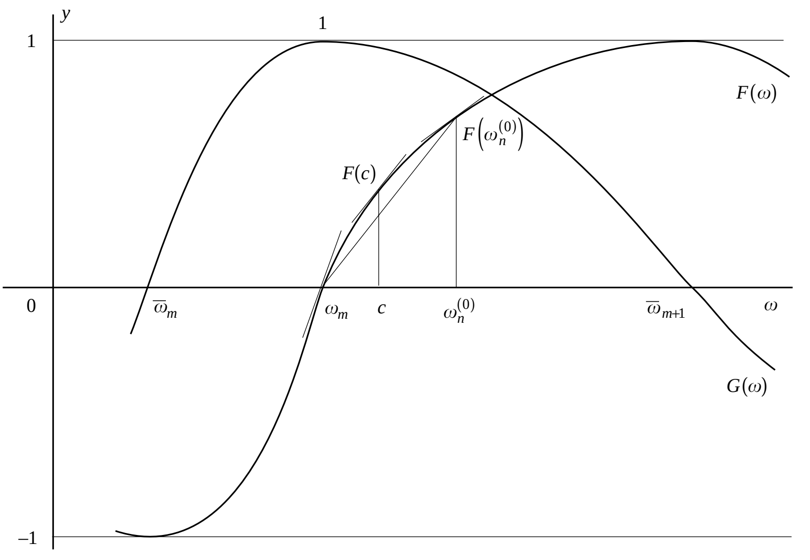

Fig. 1. The Mean Value Theorem and Concavity of $F(\omega)$ on the interval $[\omega_m, \omega_n^{(0)}], m = 2l$.

on $\omega_m < c < \omega_n^{(0)} < \varpi_{m+1}$. These properties admit a simple geometric interpretation, namely, function $|F(\omega)|$ is concave and

$$|F'(\omega_n^{(0)})| < |F'(c)| < |F'(\omega_m)|$$

on $\omega_m < c < \omega_n^{(0)} < \varpi_{m+1}$ (see Fig. 1). As a result

$$0 < \frac{|F(\omega_n^{(0)})|}{\kappa(\omega_m)} < \omega_n^{(0)} - \omega_m < \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})G(\omega_n^{(0)})} \tag{3.10}$$

by (3.7) and (3.8).

Our next step is to show that

$$\lim_{n\to\infty} \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})G(\omega_n^{(0)})} = \frac{q^{1/4}}{2(1-q^{1/2})} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} = c_1(q). \tag{3.11}$$

Indeed, by (1.1)

$$\lim_{n\to\infty} q^{-n} S_q(\eta; q^{1/4+n}) = \frac{q^{1/4}}{1-q^{1/2}} \tag{3.12}$$

and by (2.11)

$$\begin{aligned}
&\lim_{n\to\infty} q^{-n}\kappa(q^{1/4-n}) \\
&\quad = \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \lim_{n\to\infty} q^{-n} \frac{(-q^{1-2n};q)_\infty}{(-q^{1/2-2n};q)_\infty} \\
&\quad = \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \lim_{n\to\infty} q^{-n} \frac{(-q^{1-2n};q)_{2n}(-q;q)_\infty}{(-q^{1/2-2n};q)_{2n}(-q^{1/2};q)_\infty} \\
&\quad = \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \frac{(-q;q)_\infty}{(-q^{1/2};q)_\infty} \lim_{n\to\infty} \frac{(-1;q)_{2n}}{(-q^{1/2};q)_{2n}} \\
&\quad = 2\frac{(q,-q;q)_\infty^2}{(q^{1/2},-q^{1/2};q)_\infty^2} = 2\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}.
\end{aligned} \tag{3.13}$$

We have used (I.9) of [5] in the third line here. Thus, from (3.5) and (3.12)–(3.13)

$$\begin{aligned}
\lim_{n\to\infty} \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})G(\omega_n^{(0)})} &= \lim_{n\to\infty} \frac{S_q(\eta; q^{1/2}/\omega_n^{(0)})}{\kappa(\omega_n^{(0)})C_q(\eta; q^{1/2}/\omega_n^{(0)})} \\
&= \lim_{n\to\infty} \frac{q^{-n}S_q(\eta; q^{1/4+n})}{q^{-n}\kappa(q^{1/4-n})} = c_1(q).
\end{aligned}$$

Let us rewrite (3.10) as

$$0 < 1 - \frac{\omega_m}{\omega_n^{(0)}} < \frac{1}{\omega_n^{(0)}} \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})G(\omega_n^{(0)})}. \tag{3.14}$$

Taking the limit $n\to\infty$ one gets by (3.11) and the Squeeze Theorem

$$\lim_{n\to\infty}\frac{\omega_m}{\omega_n^{(0)}}=1,\quad \text{or}\quad \lim_{n\to\infty} q^n\omega_m=q^{1/4}. \tag{3.15}$$

This justifies the leading term in (3.1) (cf. [4, Theorem 5]), if one can show that $m=n$. This can be done on the basis of Jensen's Theorem and it is the only 'nonelementary' part of our proof.

The Jensen Theorem [16] states that if $f(z)$ is holomorphic in a circle of radius R with the center at the origin, and $f(0)\neq 0$, then

$$\int_0^R \frac{n_f(r)}{r}\,dr=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta-\log|f(0)|, \tag{3.16}$$

where $n_f(r)$ is the number of zeros of $f(z)$ in the circle $|z|<r$.

In view of (3.15), one can write $\omega_{m+1}<\omega_{n+1}^{(0)}<\varpi_{m+2}$. Indeed, if $\omega_m<\omega_n^{(0)}<\omega_{n+1}^{(0)}<\varpi_{m+1}$, then in a similar fashion,

$$1=\lim_{n\to\infty}\frac{\omega_m}{\omega_{n+1}^{(0)}}=\lim_{n\to\infty}\left(\frac{\omega_m}{\omega_n^{(0)}}\,\frac{\omega_n^{(0)}}{\omega_{n+1}^{(0)}}\right)=q,$$

which is a contradiction. By the definition of the $n_f(r)$ and (3.15)

$$\begin{aligned}\int_{\omega_n^{(0)}}^{\omega_{n+1}^{(0)}}\frac{n_f(r)}{r}\,dr&=\int_{\omega_n^{(0)}}^{\omega_{m+1}}\frac{n_f(r)}{r}\,dr+\int_{\omega_{m+1}}^{\omega_{n+1}^{(0)}}\frac{n_f(r)}{r}\,dr\\&=2m\log\frac{\omega_{m+1}}{\omega_n^{(0)}}+2(m+1)\log\frac{\omega_{n+1}^{(0)}}{\omega_{m+1}}\\&=2m\log\frac{\omega_{n+1}^{(0)}}{\omega_n^{(0)}}+2\log\frac{\omega_{n+1}^{(0)}}{\omega_{m+1}}\\&=2m\log q^{-1}+o(1),\quad n\to\infty,\end{aligned} \tag{3.17}$$

where $f(\omega)$ is an entire function with the simple zeros at $\omega=\pm\omega_1,\pm\omega_2,\pm\omega_3,\ldots$ defined by

$$\begin{aligned}f(\omega)&=(-q\omega^2;q^2)\frac{S_q(\eta;\omega)}{\omega}\\&=\frac{(q^{3/2}\omega^2,q^{1/2}/\omega^2;q^2)}{(q^{1/2};q)_\infty(q,-q/\omega^2;q^2)_\infty}\,C_q\left(\eta;\frac{q^{1/2}}{\omega}\right)\\&\quad+\frac{(q^{1/2}\omega^2,q^{3/2}/\omega^2;q^2)}{(q^{1/2};q)_\infty(q,-q/\omega^2;q^2)_\infty}\,S_q\left(\eta;\frac{q^{1/2}}{\omega}\right)\\&=\frac{(q^2;q^2)_\infty}{(q^{1/2};q^{1/2})_\infty}(q^{3/2}\omega^2;q^2)(1+o(1))\end{aligned} \tag{3.18}$$

by (1.1)–(1.2) as $\omega = \beta q^{-n} \to \infty$. By (3.16)

$$\int_{\omega_n^{(0)}}^{\omega_{n+1}^{(0)}} \frac{n_f(r)}{r}\, dr = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(\omega_{n+1}^{(0)} e^{i\theta})}{f(\omega_n^{(0)} e^{i\theta})} \right| d\theta \tag{3.19}$$

and in view of (3.18)

$$\begin{aligned} \frac{f(\omega_{n+1}^{(0)} e^{i\theta})}{f(\omega_n^{(0)} e^{i\theta})} &= (1 - q^{-2n} e^{2i\theta})(1 + o(1)) \\ &= -q^{-2n} e^{2i\theta}(1 + o(1)), \quad n \to \infty. \end{aligned}$$

Thus,

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(\omega_{n+1}^{(0)} e^{i\theta})}{f(\omega_n^{(0)} e^{i\theta})} \right| d\theta = 2n \log q^{-1} + o(1), \quad n \to \infty \tag{3.20}$$

and, finally, by (3.17) and (3.19)–(3.20) we obtain

$$m = n + o(1), \quad n \to \infty, \tag{3.21}$$

which implies that $m = n$ for sufficiently large n because m and n are both integers.

In order to complete the proof of the theorem one can now rewrite (3.10) with $m = n$ as

$$\begin{aligned} &|G(\omega_n^{(0)})| \frac{\kappa(\omega_n^{(0)})}{\kappa(\omega_n)} \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)}) G(\omega_n^{(0)})} \\ &\quad < \omega_n^{(0)} - \omega_n < \frac{F(\omega_n^{(0)})}{\kappa(\omega_n^{(0)}) G(\omega_n^{(0)})}. \end{aligned} \tag{3.22}$$

Due to (3.15) where $m = n$,

$$\lim_{n \to \infty} |G(\omega_n^{(0)})| = |G(\omega_n)| = 1 \tag{3.23}$$

and

$$\lim_{n \to \infty} \kappa(\omega_n^{(0)}) / \kappa(\omega_n) = 1. \tag{3.24}$$

Indeed, by (2.1) and (2.5)

$$\begin{aligned} G(\omega_n^{(0)}) &= \sqrt{\frac{(-q^{3/2-2n}; q^2)_\infty}{(-q^{1/2-2n}; q^2)_\infty}} \frac{1}{(q^{1/2}; q)_\infty} \\ &\quad \times \frac{(q^{1-2n}, q^{1+2n}; q^2)_\infty}{(q, -q^{3/2-2n}, -q^{1/2+2n}; q^2)_\infty} C_q(\eta; q^{1/4+n}). \end{aligned}$$

Using (I.9) of [5]

$$\begin{aligned}
&\lim_{n\to\infty} |G(\omega_n^{(0)})| \\
&\quad = \lim_{n\to\infty} \left(q^{n/2} \sqrt{\frac{(-q^{1/2};q^2)_n(-q^{3/2};q^2)_\infty}{(-q^{3/2};q^2)_n(-q^{1/2};q^2)_\infty}} \frac{1}{(q^{1/2};q)_\infty} \right. \\
&\qquad \left. \times q^{-n/2} \frac{(q;q^2)_n(q^{1+2n};q^2)_\infty}{(-q^{1/2};q^2)_n(-q^{1/2+2n},-q^{3/2};q^2)_\infty} C_q(\eta;q^{1/4+n}) \right) \\
&\quad = \frac{(q;q^2)_\infty}{(q^{1/2};q)_\infty(-q^{1/2},-q^{3/2};q^2)_\infty} = \frac{(q;q^2)_\infty}{(q;q^2)_\infty} = 1.
\end{aligned}$$

In a similar fashion, in view of (2.11) and (3.15) with $m = n$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\kappa(\omega_n^{(0)})}{\kappa(\omega_n)} &= \lim_{n\to\infty} \frac{(-q^{1-2n},-\omega_n^2;q)_\infty}{(-q^{1/2-2n},-q^{1/2}\omega_n^2;q)_\infty} \\
&= \frac{(-q;q)_\infty}{(-q^{1/2};q)_\infty} \lim_{n\to\infty} \frac{(-1,-q/(\omega_n^2q^{2n});q)_{2n}}{(-q^{1/2},-q^{1/2}/(\omega_n^2q^{2n});q)_{2n}} \\
&\quad \times \lim_{n\to\infty} \frac{(-(\omega_n^2q^{2n});q)_\infty}{(-q^{1/2}(\omega_n^2q^{2n});q)_\infty} \\
&= \frac{(-q,-1,-q^{1/2},-q^{1/2};q)_\infty}{(-q^{1/2},-q^{1/2},-1,-q;q)_\infty} = 1.
\end{aligned}$$

Our final step is to take the limit $n\to\infty$ in (3.22). By the Squeeze Theorem

$$\lim_{n\to\infty} (\omega_n^{(0)} - \omega_n) = c_1(q) \tag{3.25}$$

due to (3.11), (3.23) and (3.24). This proves (3.1). The asymptotic formula (3.2) can be justified in a similar fashion. We leave the details to the reader. □

Asymptotic formulas (3.1)–(3.2) can be modified in the following manner to give a better approximation for the small zeros.

Theorem 2. *Let* $0 = \omega_0 < \omega_1 < \omega_2 < \omega_3 < \cdots$ *be positive zeros of* $S_q(\eta;\omega)$ *and let* $\varpi_1 < \varpi_2 < \varpi_3 < \cdots$ *be positive zeros of* $C_q(\eta;\omega)$ *for* $0<q<1$. *Then,*

$$\omega_n = \omega_n^{(0)}\sqrt{1 - 2c_1(q)/\omega_n^{(0)}} + o(1), \tag{3.26}$$

$$\varpi_n = \varpi_n^{(0)}\sqrt{1 - 2c_1(q)/\varpi_n^{(0)}} + o(1) \tag{3.27}$$

as $n\to\infty$. *Here* $\omega_n^{(0)} = q^{1/4-n}$, $\varpi_n^{(0)} = q^{3/4-n}$, *and* $c_1(q)$ *is defined by* (3.3).

Proof. Let us consider the case of the q-sine function. Introduce

$$\omega_n^{(1)} = \omega_n^{(0)}\sqrt{1 - 2c_1(q)/\omega_n^{(0)}} \tag{3.28}$$

and rewrite (3.10) where $m=n$ in the form

$$\omega_n^{(1)} - \omega_n^{(0)} + \frac{|F(\omega_n^{(0)})|}{\kappa(\omega_n)} < \omega_n^{(1)} - \omega_n < \omega_n^{(1)} - \omega_n^{(0)} + \frac{F(\omega_n^{(0)})}{F'(\omega_n^{(0)})}. \tag{3.29}$$

Taking the limit $n \to \infty$ with the help of the same arguments as in Theorem 1 one gets

$$\lim_{n\to\infty} (\omega_n^{(1)} - \omega_n) = 0, \tag{3.30}$$

which proves (3.26). The proof of (3.27) is similar. □

Numerical analysis shows that asymptotics (3.26)–(3.27) are pretty accurate. It is of interest, nonetheless, to find next terms in asymptotic expansions (3.1)–(3.2). Numerical analysis similar to one in [7] strongly indicates that the following asymptotics hold.

Theorem 3. *Let* $0 = \omega_0 < \omega_1 < \omega_2 < \omega_3 < \cdots$ *be positive zeros of* $S_q(\eta;\omega)$ *and let* $\varpi_1 < \varpi_2 < \varpi_3 < \cdots$ *be positive zeros of* $C_q(\eta;\omega)$ *for* $0<q<1$. *Then*

$$\omega_n = \omega_n^{(0)} - c_1(q) - c_1^2(q)/(2\omega_n^{(0)}) + O(1/(\omega_n^{(0)})^2), \tag{3.31}$$

$$\varpi_n = \varpi_n^{(0)} - c_1(q) - c_1^2(q)/(2\varpi_n^{(0)}) + O(1/(\varpi_n^{(0)})^2) \tag{3.32}$$

as $n\to\infty$. *Here* $\omega_n^{(0)} = q^{1/4-n}$, $\varpi_n^{(0)} = q^{3/4-n}$, *and* $c_1(q)$ *is defined by* (3.3).

These asymptotics formally appear also if one expands the first terms in (3.26)–(3.27). This observation proves our next result.

Theorem 4. *The symbols* $o(1)$ *in* (3.26)–(3.27) *should be replaced by* $o(1/\omega_n^{(0)})$ *and* $o(1/\varpi_n^{(0)})$, *respectively*.

It does not look that there are simple proofs of these theorems by the methods of elementary calculus. We shall derive these and other improved asymptotics in Section 7; see Theorem 7 for the main result of this paper.

4. Lower and upper bounds

The following theorem provides monotone convergent sequences of the lower and upper bounds for all positive zeros of the basic sine and cosine functions.

Theorem 5. *Let* $\{\omega_n\}_{n=1}^{\infty}$ *be positive zeros of* $S_q(\eta;\omega)$ *and* $\{\varpi_n\}_{n=1}^{\infty}$ *be positive zeros of* $C_q(\eta;\omega)$ *arranged in ascending order of magnitude. Choose* $\omega_n < U_n^{(0)} < \varpi_{n+1}$,

$$L_n^{(k)} = U_n^{(k-1)} - \frac{F(U_n^{(k-1)})}{F'(U_n^{(k-1)})}, \tag{4.1}$$

$$U_n^{(k)} = U_n^{(k-1)} - \frac{|F(U_n^{(k-1)})|}{\kappa(L_n^{(k)})}, \tag{4.2}$$

and $\varpi_n < \bar{U}_n^{(0)} < \omega_n$,

$$\bar{L}_n^{(k)} = \bar{U}_n^{(k-1)} - \frac{G(\bar{U}_n^{(k-1)})}{G'(\bar{U}_n^{(k-1)})}, \tag{4.3}$$

$$\bar{U}_n^{(k)} = \bar{U}_n^{(k-1)} - \frac{|G(\bar{U}_n^{(k-1)})|}{\kappa(\bar{L}_n^{(k)})} \tag{4.4}$$

for all positive integer $k = 1, 2, 3, \ldots$. *Then*

$$L_n^{(1)} < \cdots < L_n^{(k-1)} < L_n^{(k)} < \omega_n < U_n^{(k)} < U_n^{(k-1)} < \cdots < U_n^{(0)}, \tag{4.5}$$

$$\lim_{k\to\infty} L_n^{(k)} = \lim_{k\to\infty} U_n^{(k)} = \omega_n, \tag{4.6}$$

and

$$\bar{L}_n^{(1)} < \cdots < \bar{L}_n^{(k-1)} < \bar{L}_n^{(k)} < \varpi_n < \bar{U}_n^{(k)} < \bar{U}_n^{(k-1)} < \cdots < \bar{U}_n^{(0)}, \tag{4.7}$$

$$\lim_{k\to\infty} \bar{L}_n^{(k)} = \lim_{k\to\infty} \bar{U}_n^{(k)} = \varpi_n. \tag{4.8}$$

Proof. Consider the case of the basic sine function. Let $\omega \in (\omega_n, \varpi_{n+1})$ and $\xi \in (\varpi_n, \omega_n)$. The same arguments as in Section 3—see (3.8)–(3.10) with $m = n$—

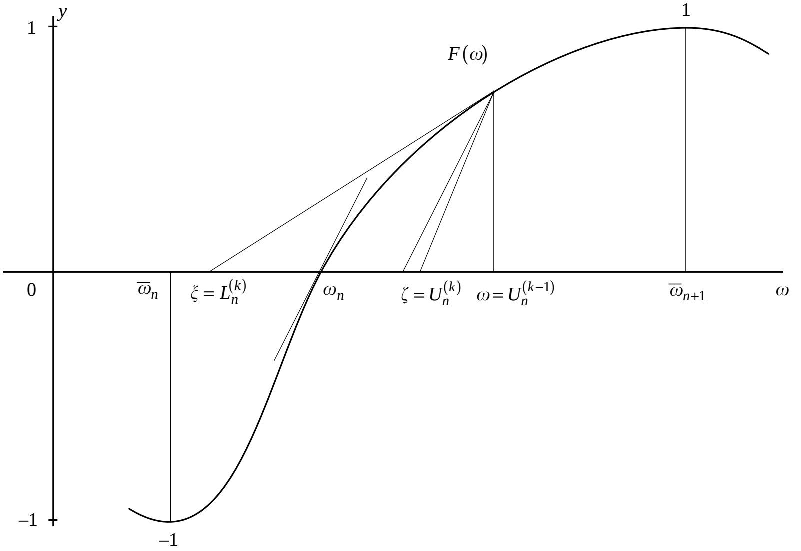

Fig. 2. Geometric interpretation of the lower $\xi = \omega - F(\omega)/F'(\omega) = L_n^{(k)}$ and upper $\zeta = \omega - |F(\omega)|/\kappa(\xi) = U_n^{(k)}$ bounds of the zero ω_n in Theorem 5, $n = 2m$.

result in

$$\frac{|F(\omega)|}{\kappa(\xi)} < \frac{|F(\omega)|}{\kappa(\omega_n)} < \omega - \omega_n < \frac{F(\omega)}{F'(\omega)}, \tag{4.9}$$

or solving for ω_n

$$\omega - \frac{F(\omega)}{F'(\omega)} < \omega_n < \omega - \frac{|F(\omega)|}{\kappa(\xi)}. \tag{4.10}$$

Substituting $\omega = U_n^{(k-1)}$ and $\xi = L_n^{(k)}$ (see Fig. 2) one gets

$$L_n^{(k)} < \omega_n < U_n^{(k)}, \quad k = 1, 2, 3, \ldots \ . \tag{4.11}$$

The monotonicity of the sequence of the upper bounds $\{U_n^{(k)}\}_{k=0}^{\infty}$ follows directly from (4.2),

$$U_n^{(k-1)} - U_n^{(k)} = \frac{|F(U_n^{(k-1)})|}{\kappa(L_n^{(k)})} > 0. \tag{4.12}$$

On the other hand, function

$$L(\omega) = \omega - \frac{F(\omega)}{F'(\omega)}, \quad L(U_n^{(k-1)}) = L_n^{(k)} \tag{4.13}$$

defined by (4.1) is monotone on (ω_n, ϖ_{n+1}) because its derivative

$$\frac{dL(\omega)}{d\omega} = \frac{F(\omega)F''(\omega)}{(F'(\omega))^2} = \frac{F(\omega)(\kappa'(\omega)G(\omega) - \kappa^2(\omega)F(\omega))}{(F'(\omega))^2} \tag{4.14}$$

does not change the sign on this interval. Hence, the sequence $\{U_n^{(k)}\}_{k=0}^{\infty}$ is decreasing and bounded below, while the $\{L_n^{(k)}\}_{k=1}^{\infty}$ is increasing and bounded above. By the Monotone Convergence Theorem the following limits exist:

$$\lim_{k\to\infty} L_n^{(k)} = L_n \leqslant \omega_n, \quad \lim_{k\to\infty} U_n^{(k)} = U_n \geqslant \omega_n. \tag{4.15}$$

Finally, taking the limit $k \to \infty$ in (4.1)–(4.2) one gets

$$L_n = U_n = \omega_n. \tag{4.16}$$

This proves (4.5)–(4.6). Similar arguments hold for the case of the basic cosine function. We leave the details to the reader. □

Construction the lower and upper bounds for the zeros of the basic sine and cosine functions in Theorem 5 is based on a simple geometric principle similar to geometric interpretation of Newton's method (see Fig. 3). Notice if, e.g., $L_n^{(k)}$ and $U_n^{(k-1)}$ satisfy (4.1), then $L_n^{(k)}$ is the ω-intercept of the tangent line to $y = F(\omega)$ at the point $\left(U_n^{(k-1)}, F\left(U_n^{(k-1)}\right)\right)$. Also, notice that $U_n^{(k)}$ defined by (4.2) is the ω-intercept of the line passing through the same point $\left(U_n^{(k-1)}, F\left(U_n^{(k-1)}\right)\right)$ with the slope

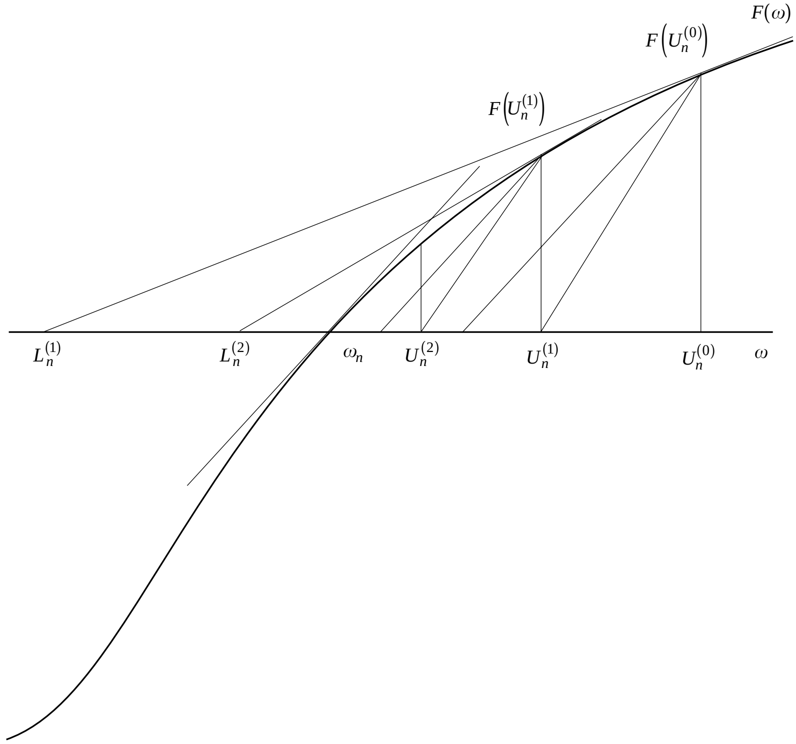

Fig. 3. First upper $U_n^{(k)}$ and lower $L_n^{(k)}$ bounds of the zero ω_n in Theorem 5, $n = 2m$.

$(-1)^n \kappa(L_n^{(k)})$, such that $\kappa(L_n^{(k)}) > \kappa(\omega_n) = |F'(\omega_n)|$. Eqs. (4.3)–(4.4) admit a similar geometric interpretation.

If the convexity of $F(\omega)$ does not change on the interval $\left(L_n^{(1)}, U_n^{(0)}\right)$ we can choose the ω-intercepts of the chords passing through the points $\left(L_n^{(k)}, F\left(L_n^{(k)}\right)\right)$ and $\left(U_n^{(k-1)}, F\left(U_n^{(k-1)}\right)\right)$ as another sequence of the upper bounds (see Fig. 4). This consideration leads to the following theorem.

Theorem 6. *Let* $\{\omega_n\}_{n=1}^{\infty}$ *be positive zeros of* $S_q(\eta;\omega)$ *and* $\{\varpi_n\}_{n=1}^{\infty}$ *be positive zeros of* $C_q(\eta;\omega)$ *arranged in ascending order of magnitude. Choose* $\omega_n < U_n^{(0)} < \varpi_{n+1}$,

$$L_n^{(k)} = U_n^{(k-1)} - \frac{F(U_n^{(k-1)})}{F'(U_n^{(k-1)})}, \tag{4.17}$$

$$U_n^{(k)} = U_n^{(k-1)} - F(U_n^{(k-1)}) \frac{U_n^{(k-1)} - L_n^{(k)}}{F(U_n^{(k-1)}) - F(L_n^{(k)})}, \tag{4.18}$$

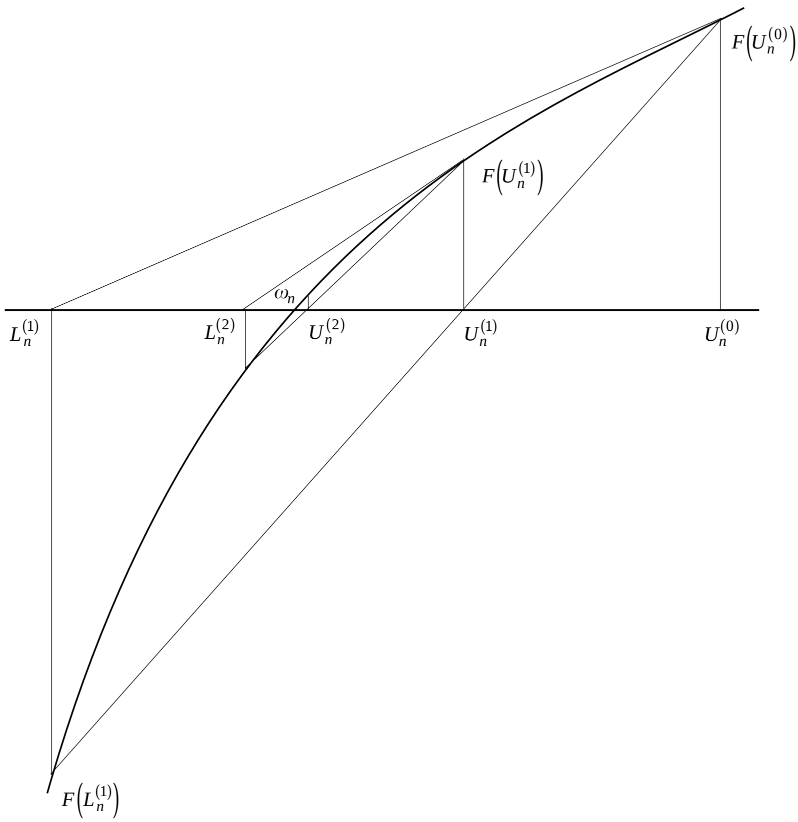

Fig. 4. First upper $U_n^{(k)}$ and lower $L_n^{(k)}$ bounds of the zero ω_n in Theorem 6, $n = 2m$.

and $\varpi_n < \bar{U}_n^{(0)} < \omega_n$,

$$\bar{L}_n^{(k)} = \bar{U}_n^{(k-1)} - \frac{G\left(\bar{U}_n^{(k-1)}\right)}{G'\left(\bar{U}_n^{(k-1)}\right)}, \tag{4.19}$$

$$\bar{U}_n^{(k)} = \bar{U}_n^{(k-1)} - G\left(\bar{U}_n^{(k-1)}\right)\frac{\bar{U}_n^{(k-1)} - \bar{L}_n^{(k)}}{G\left(\bar{U}_n^{(k-1)}\right) - G\left(\bar{L}_n^{(k)}\right)} \tag{4.20}$$

for $k = 1, 2, 3, \ldots$. *If* $F''(L_n^{(1)})F''(U_n^{(0)}) > 0$, *then*

$$L_n^{(1)} < \cdots < L_n^{(k-1)} < L_n^{(k)} < \omega_n < U_n^{(k)} < U_n^{(k-1)} < \cdots < U_n^{(0)}, \tag{4.21}$$

$$\lim_{k\to\infty} L_n^{(k)} = \lim_{k\to\infty} U_n^{(k)} = \omega_n, \tag{4.22}$$

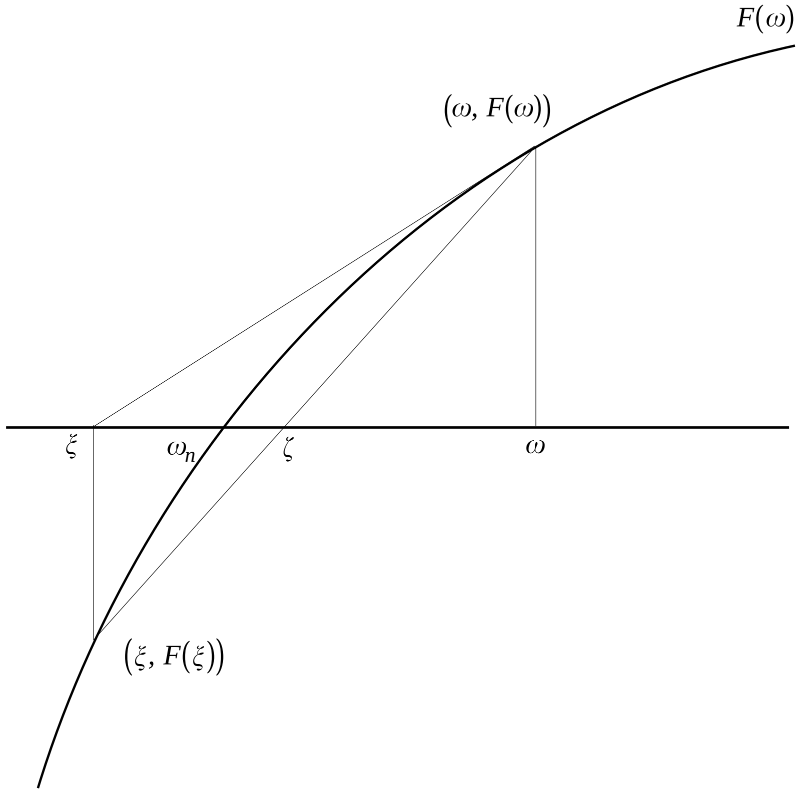

Fig. 5. Geometric interpretation of the lower $\xi = \omega - F(\omega)/F'(\omega)$ and upper $\zeta = \omega - F(\omega)(\omega - \xi)/(F(\omega) - F(\xi))$ bounds of the zero ω_n in Theorem 6, $n = 2m$.

and if $G''(L_n^{(1)})G''(U_n^{(0)}) > 0$, *then*

$$\bar{L}_n^{(1)} < \cdots < \bar{L}_n^{(k-1)} < \bar{L}_n^{(k)} < \varpi_n < \bar{U}_n^{(k)} < \bar{U}_n^{(k-1)} < \cdots < \bar{U}_n^{(0)}, \tag{4.23}$$

$$\lim_{k\to\infty} \bar{L}_n^{(k)} = \lim_{k\to\infty} \bar{U}_n^{(k)} = \varpi_n. \tag{4.24}$$

Proof. We supply the details of the proof only for the case of the basic sine function. The proof for the q-cosine function is similar. One can replace (4.10) by

$$\omega - \frac{F(\omega)}{F'(\omega)} < \omega_n < \omega - F(\omega)\frac{\omega - \xi}{F(\omega) - F(\xi)}, \tag{4.25}$$

when $\omega \in (\omega_n, \varpi_{n+1})$, $\xi \in (\varpi_n, \omega_n)$ and $F''(\xi)F''(\omega) > 0$. The second inequality holds due to the convexity of the basic sine function. Consider, for example, the case of even zeros $\omega_n = \omega_{2m}$ when F is concave on (ξ, ω) and $F'' < 0$ (see Fig. 5). The case of odd zeros can be discussed in a similar fashion. By the definition of concave function

$$F(c) > \frac{F(\omega) - F(\xi)}{\omega - \xi}(c - \omega) + F(\omega), \quad c \in (\xi, \omega), \tag{4.26}$$

which means that the chord through the points $(\xi, F(\xi))$ and $(\omega, F(\omega))$ lies below the graph of F on (ξ, ω). The ω-intercept of this chord is

$$\zeta = \omega - F(\omega)\frac{\omega - \xi}{F(\omega) - F(\xi)} \tag{4.27}$$

and

$$F(\zeta) > 0, \qquad \zeta > \omega_n, \tag{4.28}$$

which justifies the second inequality in (4.25).

Substituting $\omega = U_n^{(k-1)}$ and $\xi = L_n^{(k)}$ in (4.25) we get

$$L_n^{(k)} < \omega_n < U_n^{(k)}, \quad k = 1, 2, 3, \ldots \ . \tag{4.29}$$

The monotonicity property (4.21) can be justified in the same manner as in the proof of Theorem 5. Taking the limit $k \to \infty$ in (4.17)–(4.18) one gets (4.22). We leave the details to the reader. □

Numerical examples of the monotone sequences of the lower and upper bounds for the zeros of basic sine and cosine functions constructed in Theorems 5 and 6 are presented in Appendix F of [26], where we consider only the cases of the first zeros in order to compare the results with the corresponding sequences of the lower and upper bounds for the first zeros available from the Euler–Rayleigh method in [7]; the convergence is up to three or four times faster then one in the Euler–Rayleigh method.

5. Points of inflection

The graphs of the $F(\omega)$ and $G(\omega)$ presented in [7] show that the concavity of these q-sine and cosine functions changes somewhere before the zeros ω_n and ϖ_n, respectively. Here we discuss briefly the corresponding points of inflection. Detailed analysis and numerical investigation will appear somewhere else. Consider the case of the basic sine function $F(\omega)$. In view of (2.8),

$$F''(\omega) = \kappa'(\omega)G(\omega) - \kappa^2(\omega)F(\omega), \tag{5.1}$$

and the location of the points of inflection can be found as the roots of the equation

$$\frac{F(\omega)}{G(\omega)} = \frac{\kappa'(\omega)}{\kappa^2(\omega)} = -\left(\frac{1}{\kappa(\omega)}\right)'. \tag{5.2}$$

Also

$$\frac{d}{d\omega}\left(\frac{F(\omega)}{G(\omega)}\right) = \frac{\kappa(\omega)}{G^2(\omega)} > 0 \tag{5.3}$$

and the q-tangent function $F(\omega)/G(\omega)$ is increasing from $-\infty$ to $+\infty$ on the each interval (ϖ_n, ϖ_{n+1}). On the other hand, function $\kappa'(\omega)/\kappa^2(\omega)$ is negative and bounded for all $\omega > 0$. Thus, Eq. (5.2) has at least one solution on the each of the subinterval (ϖ_n, ω_n).

A similar analysis shows that the points of inflection of the basic cosine function $G(\omega)$ are determined as solutions of

$$\frac{G(\omega)}{F(\omega)} = -\frac{\kappa'(\omega)}{\kappa^2(\omega)} = \left(\frac{1}{\kappa(\omega)}\right)' \tag{5.4}$$

and this equation has at least one root on the each interval (ω_{n-1}, ϖ_n).

Let us consider "asymptotic solutions" of (5.2) in the form

$$\omega = \beta q^{-n}, \quad q^{3/4} < \beta < q^{1/4}. \tag{5.5}$$

Substitute (5.5) into (5.2) and take the limit $n \to \infty$. Using the limits

$$\lim_{n\to\infty} \frac{F(\beta q^{-n})}{G(\beta q^{-n})} = \beta \frac{(q^{3/2}\beta^2, q^{1/2}/\beta^2; q^2)_\infty}{(q^{1/2}\beta^2, q^{3/2}/\beta^2; q^2)_\infty} \tag{5.6}$$

and

$$\begin{aligned} &-\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\beta \lim_{n\to\infty} \frac{\kappa'(\beta q^{-n})}{\kappa^2(\beta q^{-n})} \\ &\quad = \frac{(-\beta^2, -q/\beta^2; q)_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty} \\ &\qquad + (q^{1/2};q)_\infty^2 \ \beta^2 \frac{(q^{1/2}, q^{1/2}, q^{1/2}\beta^2, 1/\beta^2; q^{1/2})_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty^2}, \end{aligned} \tag{5.7}$$

which follow from (2.1)–(2.2) and (2.11)–(2.12) in a similar fashion as in Section 3, we obtain the following transcendental equation:

$$\begin{aligned} &-\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\beta^2 \frac{(q^{3/2}\beta^2, q^{1/2}/\beta^2; q^2)_\infty}{(q^{1/2}\beta^2, q^{3/2}/\beta^2; q^2)_\infty} \\ &\quad = \frac{(-\beta^2, -q/\beta^2; q)_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty} \\ &\qquad + (q^{1/2};q)_\infty^2 \beta^2 \frac{(q^{1/2}, q^{1/2}, q^{1/2}\beta^2, 1/\beta^2; q^{1/2})_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty^2} \end{aligned} \tag{5.8}$$

for the parameter β. One can easily see that this equation has at least one solution for $q^{3/4} < \beta < q^{1/4}$.

Similar arguments hold in the case of the q-cosine function. The corresponding "asymptotic equation" has the form

$$\begin{aligned} &\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\frac{(q^{1/2}\beta^2, q^{3/2}/\beta^2; q^2)_\infty}{(q^{3/2}\beta^2, q^{1/2}/\beta^2; q^2)_\infty} \\ &\quad = \frac{(-\beta^2, -q/\beta^2; q)_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty} \\ &\qquad + (q^{1/2};q)_\infty^2\ \beta^2 \frac{(q^{1/2}, q^{1/2}, q^{1/2}\beta^2, 1/\beta^2; q^{1/2})_\infty}{(-q^{1/2}\beta^2, -q^{1/2}/\beta^2; q)_\infty^2}. \end{aligned} \tag{5.9}$$

This consideration motivates the following conjecture.

Conjecture 1. *The leading terms of the large asymptotics of the points of inflection χ_n and $\bar{\chi}_n$ of the basic sine $F(\omega)$ and cosine $G(\omega)$ functions, respectively, are defined by*

$$\lim_{n\to\infty} q^n \chi_n = \beta_0, \quad q^{3/4} < \beta_0 < q^{1/4} \tag{5.10}$$

and

$$\lim_{n\to\infty} q^n \bar{\chi}_n = \bar{\beta}_0, \quad q^{1/4} < \bar{\beta}_0 < q^{-1/4}. \tag{5.11}$$

Here β_0 and $\bar{\beta}_0$ are solutions of the transcendental equations (5.8) *and* (5.9), *respectively.*

6. Interpretation of zeros and other results

The following mechanical interpretation of the functions $F(\omega)$ and $G(\omega)$ defined by (2.4)–(2.5) can be given. Introducing a new variable

$$v(\omega) = \int_0^\omega \kappa(s)\, ds \tag{6.1}$$

one can rewrite these functions as

$$F(\omega) = \sin v(\omega), \quad G(\omega) = \cos v(\omega) \tag{6.2}$$

in view of (2.8); see also [8]. These equations describe a circular motion with the angular velocity

$$\frac{dv}{d\omega} = \kappa(\omega), \tag{6.3}$$

where the $v(\omega)$ represents the total angle of rotation as a function of "time" ω. Due to (6.1) and (6.2) the following "quantization rules" hold

$$v(\omega_n) = \int_0^{\omega_n} \kappa(s)\, ds = \pi n, \tag{6.4}$$

$$v(\varpi_n) = \int_0^{\varpi_n} \kappa(s)\, ds = \frac{\pi}{2} + \pi n \tag{6.5}$$

with $n = 0, 1, 2, \ldots$ for the nonnegative zeros ω_n and ϖ_n of the $F(\omega)$ and $G(\omega)$, respectively. This gives also a geometric interpretation of these zeros in terms of the area under the graph of function $\kappa(\omega)$. By (6.4)–(6.5) these zeros are completely defined in terms of the function $\kappa(\omega)$ given by (2.7) only.

On the other hand, one can easily verify that

$$\kappa(\omega) = \sum_{k=0}^{\infty} \frac{q^{k/2}}{1 + \omega^2 q^k} = \frac{1}{2i} \frac{d}{d\omega} \log \frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty}. \tag{6.6}$$

Therefore

$$v(\omega) = \int_0^{\omega} \kappa(s)\, ds = \frac{1}{2i} \log \frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty} \tag{6.7}$$

and our functions $F(\omega)$ and $G(\omega)$ can be rewritten as

$$\begin{aligned} F(\omega) &= \sin\left(\frac{1}{2i} \log \frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty}\right) \\ &= \frac{1}{(-\omega^2; q)_\infty^{1/2}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1/2)}}{(q^{1/2}; q^{1/2})_{2k+1}}\, \omega^{2k+1}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} G(\omega) &= \cos\left(\frac{1}{2i} \log \frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty}\right) \\ &= \frac{1}{(-\omega^2; q)_\infty^{1/2}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k-1/2)}}{(q^{1/2}; q^{1/2})_{2k}}\, \omega^{2k} \end{aligned} \tag{6.9}$$

by (1.1)–(1.2) and (2.4)–(2.5). In (6.7) we use the branch of the logarithm which takes the value $2\pi i n$ at ω_n for each of the intervals $\omega_n \leqslant \omega < \omega_{n+1}$ with the cut along the positive real axis. One can view this substitution as a q-analog of the logarithmic scale.

It is important for the further consideration to find the large ω-asymptotics of the "phase function" $v(\omega)$. Introducing the basic exponential function

$$\mathscr{E}_q(\eta; i\omega) = C_q(\eta; \omega) + iS_q(\eta; \omega) = \frac{(-i\omega; q^{1/2})_\infty}{(-q\omega^2; q^2)_\infty}, \tag{6.10}$$

see, for example, [22]; one can rewrite transformations (2.1)–(2.2) as

$$\begin{aligned} &\mathscr{E}_q(\eta; i\omega) \\ &\quad = \frac{(q^2; q^2)_\infty}{(q^{1/2}; q^{1/2})_\infty} \frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty + i\omega (q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(-q\omega^2, -q/\omega^2; q^2)_\infty} \\ &\qquad \times \mathscr{E}_q\left(\eta; \frac{iq^{1/2}}{\omega}\right). \end{aligned} \tag{6.11}$$

The expression in the second line here can be viewed as an analog of the q-exponential function corresponding to the q-trigonometric functions discussed in [6].

The last equation can be rewritten as

$$(-i\omega; q^{1/2})_\infty = \frac{(q^2; q^2)_\infty}{(q^{1/2}; q^{1/2})_\infty} \times \frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty + i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(iq^{1/2}/\omega; q^{1/2})_\infty}, \tag{6.12}$$

which gives the asymptotic behavior of the q-shifted factorials for the large values of the ω. Therefore

$$\begin{aligned} e^{2iv(\omega)} &= \frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty} \\ &= \frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty + i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty - i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty} \\ &\quad \times \frac{(-iq^{1/2}/\omega; q^{1/2})_\infty}{(iq^{1/2}/\omega; q^{1/2})_\infty} \end{aligned} \tag{6.13}$$

and the following "phase transformation" determines the large ω-asymptotic of the function $v(\omega)$:

$$\begin{aligned} v(\omega) &= \frac{1}{2i}\log\frac{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty + i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty}{(q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty - i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty} \\ &\quad + v(q^{1/2}/\omega). \end{aligned} \tag{6.14}$$

In view of (6.3) and (6.12) this implies

$$\begin{aligned} \kappa(\omega) &= \frac{1}{2i}\frac{d}{d\omega}\log\frac{(-i\omega, iq^{1/2}/\omega; q^{1/2})_\infty}{(i\omega, -iq^{1/2}/\omega; q^{1/2})_\infty} \\ &\quad - \frac{q^{1/2}}{\omega^2}\kappa\left(\frac{q^{1/2}}{\omega}\right). \end{aligned} \tag{6.15}$$

Comparing this transformation with (2.11), we arrive at the following differentiation formula:

$$\begin{aligned} &\frac{1}{2i}\frac{d}{d\omega}\log\frac{(-i\omega, iq^{1/2}/\omega; q^{1/2})_\infty}{(i\omega, -iq^{1/2}/\omega; q^{1/2})_\infty} \\ &\quad = \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\frac{(-q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{(-\omega^2, -q/\omega^2; q)_\infty} \end{aligned} \tag{6.16}$$

and at the corresponding indefinite integral

$$\begin{aligned} &\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\int \frac{(-q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{(-\omega^2, -q/\omega^2; q)_\infty}\, d\omega \\ &\quad = \frac{1}{2i}\log\frac{(-i\omega, iq^{1/2}/\omega; q^{1/2})_\infty}{(i\omega, -iq^{1/2}/\omega; q^{1/2})_\infty}. \end{aligned} \tag{6.17}$$

Use of (6.6) gives an independent proof of these relations

$$\begin{aligned}
&\frac{d}{d\omega}\log\frac{(-i\omega, iq^{1/2}/\omega; q^{1/2})_\infty}{(i\omega, -iq^{1/2}/\omega; q^{1/2})_\infty}\\
&\quad=\frac{d}{d\omega}\log\frac{(-i\omega; q^{1/2})_\infty}{(i\omega; q^{1/2})_\infty}-\frac{d}{d\omega}\log\frac{(-iq^{1/2}/\omega; q^{1/2})_\infty}{(iq^{1/2}/\omega; q^{1/2})_\infty}\\
&\quad=2i\left(\kappa(\omega)+\frac{q^{1/2}}{\omega^2}\kappa\left(\frac{q^{1/2}}{\omega}\right)\right)=2i\sum_{k=-\infty}^{\infty}\frac{q^{k/2}}{1+\omega^2q^k}\\
&\quad=2i\,\frac{(q, q, -q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{(q^{1/2}, q^{1/2}, -\omega^2, -q/\omega^2; q)_\infty}
\end{aligned}\tag{6.18}$$

by the Ramanujan ${}_1\psi_1$ summation formula (cf. [24, Appendix 12.2]).

In the next section we shall derive new asymptotic formulas which improve (1.3)–(1.4) and (3.25)–(3.26) with the help of (6.4)–(6.5) and (6.14). This will be done by means of the inverse function of the "phase function" $v(\omega)$, which can be found by the Lagrange inversion formula. To make our paper as self-contained as possible, we remind the reader this formula; see, for example, [1,2,17] and Appendix B.2 of [26] for more details.

Let $w = g(z)$ be regular at $z = z_0$:

$$w = g(z) = w_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n, \quad a_1 = g'(z_0) \neq 0. \tag{6.19}$$

Then the inverse function $z = g^{-1}(w) = h(g(z))$ can be found by the Lagrange inversion formula as

$$z = h(w) = z_0 + \sum_{n=1}^{\infty} b_n (w - w_0)^n, \tag{6.20}$$

where

$$\begin{aligned}
b_n &= \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{zg'(z)}{(g(z)-w_0)^{n+1}}\,dz\\
&=\frac{1}{n!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z-z_0}{g(z)-w_0}\right)^n,
\end{aligned}\tag{6.21}$$

see, [2,17] for the proof. Moreover, the first coefficients of expansion (6.20) are given explicitly by

$$b_1 = \frac{1}{a_1}, \quad b_2 = -\frac{a_2}{a_1^3}, \quad b_3 = \frac{1}{a_1^3}\left(2\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right). \tag{6.22}$$

We introduce the inverse of the $v(\omega)$ as the power series

$$\omega = f(v) = \omega_0 + \sum_{k=1}^{\infty} b_k (v - v_0)^k \tag{6.23}$$

with the coefficients

$$
\begin{aligned}
b_k &= \frac{f^{(k)}(v_0)}{k!} = \frac{1}{2\pi i}\int_{|\omega-\omega_0|=r} \frac{\omega\kappa(\omega)}{(v(\omega)-v_0)^{k+1}}\,d\omega \\
&= \frac{1}{k!}\lim_{\omega\to\omega_0}\frac{d^{k-1}}{d\omega^{k-1}}\left(\frac{\omega-\omega_0}{v(\omega)-v_0}\right)^k
\end{aligned}
\tag{6.24}
$$

in view of (6.20)–(6.21) and (6.3). The first terms of this Lagrange expansion are

$$
\begin{aligned}
\omega = \omega_0 &+ \frac{1}{\kappa(\omega_0)}(v-v_0) - \frac{\kappa'(\omega_0)}{2\kappa^3(\omega_0)}(v-v_0)^2 \\
&+ \frac{1}{2\kappa^3(\omega_0)}\left(\left(\frac{\kappa'(\omega_0)}{\kappa(\omega_0)}\right)^2 - \frac{\kappa''(\omega_0)}{3\kappa(\omega_0)}\right)(v-v_0)^3 + \cdots
\end{aligned}
\tag{6.25}
$$

due to (6.22).

Remark 1. Relation (6.12) can be rewritten as

$$
\begin{aligned}
&\frac{(q^{1/2};q^{1/2})_\infty}{(q^2;q^2)_\infty}(-i\omega, iq^{1/2}/\omega; q^{1/2})_\infty \\
&\quad = (q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)_\infty + i\omega(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)_\infty,
\end{aligned}
\tag{6.26}
$$

which implies another identity

$$
\begin{aligned}
&\frac{(q^{1/2};q^{1/2})^2_\infty}{(q^2;q^2)^2_\infty}(-\omega^2, -q/\omega^2; q)_\infty \\
&\quad = (q^{1/2}\omega^2, q^{3/2}/\omega^2; q^2)^2_\infty + \omega^2(q^{3/2}\omega^2, q^{1/2}/\omega^2; q^2)^2_\infty.
\end{aligned}
\tag{6.27}
$$

Eq. (6.26) can also be derived as a special case of the Exercise 5.21 of [5].

Remark 2. Let us notice also how the limiting case $q\to 1^-$ of Eq. (6.8)–(6.9) gives the classical sine and cosine functions. By (6.7)

$$
v(\omega) = \frac{1}{2i}\log\frac{(-i\omega;q^{1/2})_\infty}{(i\omega;q^{1/2})_\infty} = \frac{1}{2i}\log(e_{q^{1/2}}(i\omega)\ E_{q^{1/2}}(i\omega)),
\tag{6.28}
$$

where the Jackson q-exponential functions,

$$
e_q(z) = \sum_{n=0}^\infty \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z|<1,
\tag{6.29}
$$

$$
E_q(z) = \sum_{n=0}^\infty q^{n(n-1)/2}\frac{z^n}{(q;q)_n} = (-z;q)_\infty,
\tag{6.30}
$$

have the limits

$$
\lim_{q\to 1^-} e_q((1-q)z) = \lim_{q\to 1^-} E_q((1-q)z) = e^z,
$$

see, for example, [5]. Thus

$$\begin{aligned}\lim_{q\to 1^-} & v((1-q^{1/2})\omega)\\ &= \frac{1}{2i}\log\left(\lim_{q\to 1^-} e_{q^{1/2}}(i(1-q^{1/2})\omega)E_{q^{1/2}}(i(1-q^{1/2})\omega)\right)\\ &= \frac{1}{2i}\log(\exp(2i\omega)) = \omega \end{aligned} \tag{6.31}$$

and

$$\lim_{q\to 1^-} F((1-q^{1/2})\omega) = \sin\omega, \quad \lim_{q\to 1^-} G((1-q^{1/2})\omega) = \cos\omega. \tag{6.32}$$

Remark 3. In a similar fashion, one can see that

$$\begin{aligned}\frac{(q;q)_n}{(1-q)^n} &= 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})\\ &= n!\left(1-\frac{n(n-1)}{4}(1-q)+O((1-q)^2)\right)\end{aligned} \tag{6.33}$$

as $q\to 1^-$. This means, formally,

$$e_q((1-q)z) = e^z\left(1+\frac{1}{4}(1-q)z^2\right) + O((1-q)^2), \tag{6.34}$$

$$E_q((1-q)z) = e^z\left(1-\frac{1}{4}(1-q)z^2\right) + O((1-q)^2) \tag{6.35}$$

and

$$e_q((1-q)z)E_q((1-q)z) = e^{2z} + O((1-q)^2), \quad q\to 1^-. \tag{6.36}$$

Thus we obtain

$$\begin{aligned} v(&(1-q^{1/2})\omega)\\ &= \frac{1}{2i}\log(e_{q^{1/2}}(i(1-q^{1/2})\omega)E_{q^{1/2}}(i(1-q^{1/2})\omega))\\ &= \frac{1}{2i}\log(\exp(2i\omega)(1+O((1-q^{1/2})^2)))\\ &= \omega + O((1-q^{1/2})^2),\end{aligned} \tag{6.37}$$

formally, as $q\to 1^-$. In view of (6.4)–(6.5), this automatically leads us to the following asymptotic formulas:

$$\omega_n = \pi n(1-q^{1/2}) + O((1-q^{1/2})^3), \tag{6.38}$$

$$\varpi_n = \left(\frac{\pi}{2}+\pi n\right)(1-q^{1/2}) + O((1-q^{1/2})^3) \tag{6.39}$$

valid for small zeros of the basic sine and cosine functions, respectively, in the limit $q\to 1^-$. It has been suggested by the referee that these asymptotics can be obtained by applying Sturm–Liouville theory to Eq. (2.9).

7. Improved asymptotics

This section contains the main result of this paper. It is written in response to the referee's suggestion to establish a few more terms in asymptotic expansions (1.3)–(1.4) for the q-sine and cosine functions under consideration. Before seeing the referee's report the author was able independently to prove formulas (3.31)–(3.32), which have been originally conjectured in the first version of this paper, and to find one more term by a completely different method mused, partially, by the referee. Similar idea can be applied in order to establish next terms of these asymptotic expansions if needed. These are our findings.

In principle, Eqs. (6.4)–(6.5) give explicit formulas for the zeros ω_n and ϖ_n in terms of the inverse function (6.23), namely,

$$\omega_n = f(\pi n), \quad \varpi_n = f\left(\frac{\pi}{2} + \pi n\right). \tag{7.1}$$

The problem is to find a convenient representation for the $f(v)$ valid for the large ω.

In the case of the basic sine function, consider $\omega_0 = \omega_n^{(0)} = q^{1/4-n}$. In view of (6.14) we get

$$v_n^{(0)} = v(\omega_n^{(0)}) = \pi n + v\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) \tag{7.2}$$

and by (6.25)

$$\begin{aligned} \omega_n &= f(\pi n) = f\left(v_n^{(0)} - v\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right)\right) \\ &= \omega_n^{(0)} - \frac{1}{\kappa(\omega_n^{(0)})} v\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) - \frac{\kappa'(\omega_n^{(0)})}{2\kappa^3(\omega_n^{(0)})} v^2\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) \\ &\quad - \frac{1}{2\kappa^3(\omega_n^{(0)})}\left(\left(\frac{\kappa'(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})}\right)^2 - \frac{\kappa''(\omega_n^{(0)})}{3\kappa(\omega_n^{(0)})}\right) v^3\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) + \cdots, \end{aligned} \tag{7.3}$$

where the series converges absolutely and uniformly in a certain closed disk centered at $v_n^{(0)}$.

In order to establish an asymptotic formula for the large positive zeros ω_n of the q-sine function one can consider the large n-asymptotic of the first terms in the convergent series (7.3). This can be done with the help of the Taylor expansions

$$\kappa(\omega) = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k}}{1-q^{k+1/2}} = \frac{1}{1-q^{1/2}} - \frac{\omega^2}{1-q^{3/2}} + \frac{\omega^4}{1-q^{5/2}} - \ \cdots\ , \tag{7.4}$$

$$\begin{aligned} v(\omega) &= \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k+1}}{(2k+1)(1-q^{k+1/2})} \\ &= \frac{\omega}{1-q^{1/2}} - \frac{\omega^3}{3(1-q^{3/2})} + \frac{\omega^5}{5(1-q^{5/2})} - \cdots \end{aligned} \tag{7.5}$$

valid for $|\omega|<1$ as easy consequences of (6.1) and (6.6) and with the help of the following lemma.

Lemma 1. *The following asymptotics hold*:

$$\frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2}\frac{(-q^{1/2}(\omega_n^{(0)})^2,-q^{1/2}/(\omega_n^{(0)})^2;q)_\infty}{(-(\omega_n^{(0)})^2,-q/(\omega_n^{(0)})^2;q)_\infty} = \frac{q^{1/2}}{1-q^{1/2}}c_1^{-1}(q)(\omega_n^{(0)})^{-1}(1+O((\omega_n^{(0)})^{-4})), \tag{7.6}$$

$$\kappa(\omega_n^{(0)}) = \frac{q^{1/2}}{1-q^{1/2}}c_1^{-1}(q)(\omega_n^{(0)})^{-1} \times\left(1-\frac{c_1(q)}{\omega_n^{(0)}}+\frac{q}{1+q^{1/2}+q}\frac{c_1(q)}{(\omega_n^{(0)})^3}+O((\omega_n^{(0)})^{-4})\right), \tag{7.7}$$

$$\kappa'(\omega_n^{(0)}) = -\frac{q^{1/2}}{1-q^{1/2}}c_1^{-1}(q)(\omega_n^{(0)})^{-2} \times\left(1-\frac{2c_1(q)}{\omega_n^{(0)}}+\frac{4q}{1+q^{1/2}+q}\frac{c_1(q)}{(\omega_n^{(0)})^3}+O((\omega_n^{(0)})^{-4})\right), \tag{7.8}$$

$$\kappa''(\omega_n^{(0)}) = \frac{q^{1/2}}{1-q^{1/2}}\frac{\alpha(q)}{c_1(q)}(\omega_n^{(0)})^{-3} \times\left(1-\frac{6c_1(q)}{\alpha(q)}\frac{1}{\omega_n^{(0)}}+O((\omega_n^{(0)})^{-3})\right) \tag{7.9}$$

as $n\to\infty$. *Here* $\omega_n^{(0)}=q^{1/4-n}$, $0<q<1$; $c_1(q)$ *is defined by* (3.3), *and*

$$\alpha(q) = 3-2\frac{(q^{1/2};q)_\infty^4\,(q^2;q^2)_\infty^2}{(q;q)_\infty^4\,(q;q^2)_\infty^2}\left(1+2^4\sum_{k=1}^{\infty}\frac{q^{3k/2}}{(1+q^k)^3}\right) + 2\frac{(q^{1/2};q^{1/2})_\infty^4}{(-q^{1/2};q^{1/2})_\infty^4}. \tag{7.10}$$

The same asymptotics are valid in the case $\varpi_n^{(0)}=q^{3/4-n}$.

Proof. By the q-binomial formula

$$\frac{(-q^{1/2}/\omega^2;q)_\infty}{(-q/\omega^2;q)_\infty} = \sum_{k=0}^{\infty}\frac{(q^{-1/2};q)_k}{(q;q)_k}\left(-\frac{q}{\omega^2}\right)^k = 1+\frac{q^{1/2}}{1+q^{1/2}}\omega^{-2}+O(\omega^{-4}),\quad |\omega|\to\infty. \tag{7.11}$$

Also

$$\frac{(-q^{1/2}(\omega_n^{(0)})^2;q)_\infty}{(-(\omega_n^{(0)})^2;q)_\infty} = \frac{(-q^{1-2n};q)_\infty}{(-q^{1/2-2n};q)_\infty} = \frac{(-q^{1-2n};q)_{2n}(-q;q)_\infty}{(-q^{1/2-2n};q)_{2n}(-q^{1/2};q)_\infty} \tag{7.12}$$

and by (I.9) of [5] and the q-binomial theorem

$$\begin{aligned}\frac{(-q^{1-2n};q)_{2n}}{(-q^{1/2-2n};q)_{2n}} &= q^n \frac{(-1;q)_{2n}}{(-q^{1/2};q)_{2n}} = q^n \frac{(-q^{1/2+2n};q)_\infty}{(-q^{2n};q)_\infty} \frac{(-1;q)_\infty}{(-q^{1/2};q)_\infty} \\ &= q^n \frac{(-1;q)_\infty}{(-q^{1/2};q)_\infty}\left(1 - \frac{q^{2n}}{1+q^{1/2}} + O(q^{4n})\right), \quad n\to\infty.\end{aligned} \tag{7.13}$$

As a result

$$\begin{aligned}\frac{(-q^{1/2}(\omega_n^{(0)})^2;q)_\infty}{(-(\omega_n^{(0)})^2;q)_\infty} &= 2q^{1/4}\frac{(-q;q)_\infty^2}{(-q^{1/2};q)_\infty^2} \\ &\times (\omega_n^{(0)})^{-1}\left(1 - \frac{q^{1/2}}{1+q^{1/2}}(\omega_n^{(0)})^{-2} + O((\omega_n^{(0)})^{-4})\right), \quad n\to\infty.\end{aligned} \tag{7.14}$$

Combining (7.11) and (7.14), one gets (7.6) with the help of (3.3).

Eq. (7.7) follows now from (2.11), (7.4) and (7.6). In a similar manner, Eq. (7.8) follows from (2.12), where

$$\kappa_1(\omega) = \sum_{k=0}^\infty \frac{q^{k/2}}{(1+\omega^2 q^k)^2} = \frac{1}{1-q^{1/2}} - \frac{2\omega^2}{1-q^{3/2}} + O(\omega^4) \tag{7.15}$$

as $|\omega|\to 0$.

The proof of (7.9) can be given on the basis of formula (A.5) from Appendix A. By the q-binomial formula

$$\begin{aligned}\frac{(q/\omega^2;q)_\infty}{(-q/\omega^2;q)_\infty} &= \sum_{k=0}^\infty \frac{(-1;q)_k}{(q;q)_k}\left(-\frac{q}{\omega^2}\right)^k \\ &= 1 - \frac{2q}{1-q}\omega^{-2} + \frac{2q^2}{(1-q)^2}\omega^{-4} + O(\omega^{-6}), \quad |\omega|\to\infty.\end{aligned} \tag{7.16}$$

On the other hand

$$\frac{\left(\left(\omega_n^{(0)}\right)^2;q\right)_\infty}{\left(-\left(\omega_n^{(0)}\right)^2;q\right)_\infty} = \frac{(q^{1/2-2n};q)_\infty}{(-q^{1/2-2n};q)_\infty} = \frac{(q^{1/2-2n};q)_{2n}(q^{1/2};q)_\infty}{(-q^{1/2-2n};q)_{2n}(-q^{1/2};q)_\infty} \tag{7.17}$$

and by (I.9) of [5] and the q-binomial theorem

$$\begin{aligned}\frac{(q^{1/2-2n};q)_{2n}}{(-q^{1/2-2n};q)_{2n}} &= \frac{(q^{1/2};q)_{2n}}{(-q^{1/2};q)_{2n}} = \frac{(-q^{1/2+2n};q)_\infty}{(q^{1/2+2n};q)_\infty}\frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty} \\ &= \frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty}\left(1 + \frac{2q^{1/2+2n}}{1-q} + \frac{2q^{1+4n}}{(1-q)^2} + O(q^{6n})\right)\end{aligned} \tag{7.18}$$

as $n \to \infty$. Thus

$$\frac{\left(\left(\omega_n^{(0)}\right)^2; q\right)_\infty}{(-(\omega_n^{(0)})^2; q)_\infty} = \frac{(q^{1/2}; q)_\infty^2}{(-q^{1/2}; q)_\infty^2} \times \left(1 + \frac{2q}{1-q}(\omega_n^{(0)})^{-2} + \frac{2q^2}{(1-q)^2}(\omega_n^{(0)})^{-4} + O((\omega_n^{(0)})^{-6})\right) \tag{7.19}$$

as $n \to \infty$ and combining (7.16) and (7.19), one gets

$$\frac{\left(\left(\omega_n^{(0)}\right)^2, q/\left(\omega_n^{(0)}\right)^2; q\right)_\infty}{\left(-\left(\omega_n^{(0)}\right)^2, -q/\left(\omega_n^{(0)}\right)^2; q\right)_\infty} = \frac{(q^{1/2}; q)_\infty^2}{(-q^{1/2}; q)_\infty^2}\left(1 + O\left(\left(\omega_n^{(0)}\right)^{-6}\right)\right) \tag{7.20}$$

as $n \to \infty$. Now asymptotic (7.9) follows from (A.5), (7.6) and (7.20). The case $\varpi_n^{(0)} = q^{3/4-n}$ can be considered in a similar fashion. □

The new improved asymptotic formula for the ω_n can be derived now from expansion (7.3) if we substitute

$$\begin{aligned}\frac{1}{\kappa\left(\omega_n^{(0)}\right)}\, v\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) &= c_1(q) + c_1(q)^2(\omega_n^{(0)})^{-1} + c_1(q)\left(c_1^2(q) - \frac{q}{3(1+q^{1/2}+q)}\right)(\omega_n^{(0)})^{-2} \\ &\quad + c_1^2(q)\left(c_1^2(q) - \frac{4q}{3(1+q^{1/2}+q)}\right)\left(\omega_n^{(0)}\right)^{-3} + O\left(\left(\omega_n^{(0)}\right)^{-4}\right),\end{aligned} \tag{7.21}$$

$$\frac{\kappa'(\omega_n^{(0)})}{2\kappa^3(\omega_n^{(0)})} v^2\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) = -\frac{c_1^2(q)}{2\omega_n^{(0)}}\left(1 + c_1(q)(\omega_n^{(0)})^{-1} - \frac{2q}{3(1+q^{1/2}+q)}\left(\omega_n^{(0)}\right)^{-2} + O(\omega_n^{(0)})^{-3}\right) \tag{7.22}$$

and

$$\frac{1}{2\kappa^3(\omega_n^{(0)})}\left(\left(\frac{\kappa'(\omega_n^{(0)})}{\kappa(\omega_n^{(0)})}\right)^2 - \frac{\kappa''(\omega_n^{(0)})}{3\kappa(\omega_n^{(0)})}\right) v^3\left(\frac{q^{1/2}}{\omega_n^{(0)}}\right) = -\frac{c_1^3(q)(\alpha(q)-3)}{6(\omega_n^{(0)})^2}\left(1 + \frac{4\alpha(q)-9}{\alpha(q)-3} c_1(q)(\omega_n^{(0)})^{-1} + O(\omega_n^{(0)})^{-2}\right) \tag{7.23}$$

as $n \to \infty$ in view of (7.4)–(7.9).

The case of the q-cosine function can be considered in a similar fashion. We summarize our results in the following main theorem of this paper.

Theorem 7. *Let* $0=\omega_0<\omega_1<\omega_2<\omega_3<\cdots$ *be positive zeros of* $S_q(\eta;\omega)$ *and let* $\varpi_1<\varpi_2<\varpi_3<\cdots$ *be positive zeros of* $C_q(\eta;\omega)$ *for* $0<q<1$. *Then*

$$\omega_n=\omega_n^{(0)}-c_1(q)-\frac{c_1^2(q)}{2\omega_n^{(0)}} - \left(c_1^2(q)(2c_2(q)+3)-\frac{2q}{1+q^{1/2}+q}\right)\frac{c_1(q)}{6(\omega_n^{(0)})^2}+O((\omega_n^{(0)})^{-3}) \tag{7.24}$$

and

$$\varpi_n=\varpi_n^{(0)}-c_1(q)-\frac{c_1^2(q)}{2\varpi_n^{(0)}} - \left(c_1^2(q)(2c_2(q)+3)-\frac{2q}{1+q^{1/2}+q}\right)\frac{c_1(q)}{6(\varpi_n^{(0)})^2}+O((\varpi_n^{(0)})^{-3}) \tag{7.25}$$

as $n\to\infty$. *Here* $\omega_n^{(0)}=q^{1/4-n}$, $\varpi_n^{(0)}=q^{3/4-n}$, *and*

$$c_1(q)=\frac{q^{1/4}}{2(1-q^{1/2})}\frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}, \tag{7.26}$$

$$c_2(q)=\frac{(q^{1/2};q)_\infty^4\,(q^2;q^2)_\infty^2}{(q;q)_\infty^4\,(q;q^2)_\infty^2}\left(1+2^4\sum_{k=1}^{\infty}\frac{q^{3k/2}}{(1+q^k)^3}\right) - \frac{(q^{1/2};q^{1/2})_\infty^4}{(-q^{1/2};q^{1/2})_\infty^4}. \tag{7.27}$$

The functions $c_1(q)$ and $c_2(q)$ are nonnegative and increasing on $[0,1]$; see Fig. 6.

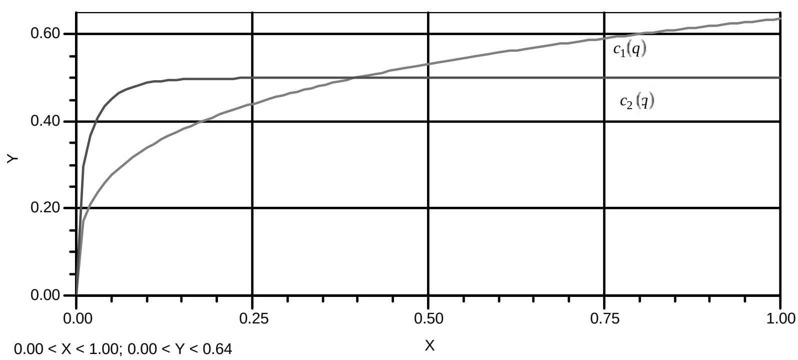

Fig. 6. Functions $c_1(q)$ and $c_2(q)$, $0\leqslant q\leqslant 1$.

Numerical examples demonstrating accuracy of these formulas are given in Appendix D of [26]; these asymptotic formulas give a very good approximation even for small zeros. See also Eqs. (B.3) and (B.5) from Appendix B here for alternative forms of the constant $c_2(q)$. The proof of the monotonicity of the $c_1(q)$ is given in Appendix C.

8. Application: analytic continuation of q-zeta function

The Riemann zeta function is

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{8.1}$$

This series converges uniformly and absolutely for $\operatorname{Re} z > 1$ and, therefore, defines a holomorphic function in the half-plane $\operatorname{Re} z > 1$. For the analytic continuation of this function in the entire complex plane, other properties and applications, see, for example, [1,27].

An extension of the zeta function has been recently introduced in [24], see also our review paper [22] and Section 10.6 of [26], as

$$\zeta_q(z) = \sum_{n=1}^{\infty} \frac{1}{\kappa(\omega_n)\omega_n^z}, \tag{8.2}$$

where $\{\omega_n\}_{n=1}^{\infty}$ are the positive zeros of the q-sine function (1.1) and the $\kappa(\omega)$ is defined by (6.6). The right side here is a uniformly and absolutely convergent series of analytic functions in any domain $\operatorname{Re} z > 1$ and consequently the series is an analytic function in such a domain. See [24] for the details. Here we give, first of all, an independent proof of this result using elementary facts about convergence of the Dirichlet series of the general form [2,17]

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \tag{8.3}$$

where the a_n are complex numbers and the exponents λ_n are nonnegative real numbers satisfying the conditions

$$\lambda_n < \lambda_{n+1}, \quad n = 1, 2, 3, \ldots, \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{8.4}$$

The q-zeta function above is the Dirichlet series with

$$a_n = \frac{1}{\kappa(\omega_n)}, \quad \lambda_n = \log \omega_n. \tag{8.5}$$

In view of

$$\omega_n = q^{1/4-n} + O(1), \quad n \to \infty \tag{8.6}$$

one gets

$$\lambda_n = \log \omega_n = (n - 1/4) \log q^{-1} + O(q^n), \quad n \to \infty \tag{8.7}$$

and both conditions (8.4) are, obviously, satisfied for $0<q<1$.

By the definition, the Dirichlet series (8.3) is said to have abscissa of convergence C, or half-plane of convergence $\operatorname{Re} z > C$, if the series converges at every point of the half-plane $\operatorname{Re} z > C$, but diverges at any point of the half-plane $\operatorname{Re} z < C$. Similarly, the Dirichlet series (8.3) is said to have abscissa of absolute convergence A, or half-plane of absolute convergence $\operatorname{Re} z > A$, if the series converges absolutely at every point of the half-plane $\operatorname{Re} z > A$, but not at any point of the half-plane $\operatorname{Re} z < A$; see [2,17] for more details. The following theorem is a generalization of the Cauchy–Hadamard Theorem for the power series.

Theorem 8. *For the Dirichlet series* (8.3) *satisfying conditions* (8.4) *and*

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0, \tag{8.8}$$

the numbers A and C are given by the formula

$$A = C = \limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n}. \tag{8.9}$$

One can look at [2,17] for the proof of this theorem.

In the case of the q-zeta function (8.2) we get

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = \lim_{n\to\infty} \frac{\log n}{n \log q^{-1} + O(1)} = \frac{1}{\log q^{-1}} \lim_{n\to\infty} \frac{\log n}{n} = 0 \tag{8.10}$$

due to (8.7) and condition (8.8) is satisfied. We shall show in this section that

$$\kappa(\omega_n) = 2\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} q^n (1 + o(1)), \quad n \to \infty, \tag{8.11}$$

see (8.14) below. Thus

$$\log|a_n| = -\log \kappa(\omega_n) = n \log q^{-1} + O(1), \quad n \to \infty \tag{8.12}$$

and by (8.7) and (8.9)

$$A = C = \lim_{n\to\infty} \frac{\log|a_n|}{\lambda_n} = \lim_{n\to\infty} \frac{n \log q^{-1} + O(1)}{n \log q^{-1} + O(1)} = 1. \tag{8.13}$$

Therefore by Theorem 8 the series (8.2) for the $\zeta_q(z)$ converges absolutely and uniformly in the half-plane $\operatorname{Re} z > 1$ and defines an analytic function in such a half-plane. This series diverges in the half-plane $\operatorname{Re} z < 1$.

Analytic continuation of the $\zeta_q(z)$ in the entire complex plane is an interesting open problem. We shall show here that, as in the classical case, the $\zeta_q(z)$ has a simple pole at $z = 1$ and it has no other singularities in the half-plane $\operatorname{Re} z > 0$. With the help of the improved asymptotic for the zeros of the basic sine function found in

Section 7, we will be able to show that, in addition, our q-zeta function has simple poles at $z=-1$ and -2 and it has no other singularities in the half-plane $\operatorname{Re} z > -3$.

We first establish the following asymptotic formula:

$$\begin{aligned}\kappa(\omega_n) =& \frac{q^{1/2}}{(1-q^{1/2})c_1(q)}(\omega_n^{(0)})^{-1} + \frac{q^{1/2}c_1(q)c_2(q)}{1-q^{1/2}}(\omega_n^{(0)})^{-3} \\ &+ \frac{q^{1/2}}{2(1-q^{1/2})}\left(c_1^2(q)(2c_2(q)-3) + \frac{2q}{1+q^{1/2}+q}\right)(\omega_n^{(0)})^{-4} \\ &+ O((\omega_n^{(0)})^{-5}), \quad n\to\infty,\end{aligned} \tag{8.14}$$

where $\omega_n^{(0)} = q^{1/4-n}$; which is of independent interest. Indeed, Taylor's formula

$$\kappa(\omega_n) = \kappa(\omega_n^{(0)}) + \kappa'(\omega_n^{(0)})(\omega_n - \omega_n^{(0)}) + \frac{1}{2}\kappa''(\omega_n^{(0)})(\omega_n - \omega_n^{(0)})^2 + \cdots \tag{8.15}$$

and asymptotics (7.7)–(7.9) and (7.24) result in (8.14) after some simplification.

Eq. (8.14) implies

$$\begin{aligned}\frac{1}{\kappa(\omega_n)} =& \frac{(1-q^{1/2})c_1(q)}{q^{1/2}}\omega_n^{(0)}\left(1 - \frac{c_1^2(q)c_2(q)}{(\omega_n^{(0)})^2}\right. \\ &\left. -\left(c_1^2(q)(2c_2(q)-3) + \frac{2q}{1+q^{1/2}+q}\right)\frac{c_1(q)}{2(\omega_n^{(0)})^3} + O((\omega_n^{(0)})^{-4})\right)\end{aligned} \tag{8.16}$$

as $n\to\infty$.

In a similar fashion, with the help of the binomial formula and (7.24)

$$\begin{aligned}\omega_n^{-z} =& (\omega_n^{(0)})^{-z}\left(1 + \frac{zc_1(q)}{\omega_n^{(0)}} + \frac{z(z+2)c_1(q)}{2(\omega_n^{(0)})^2}\right. \\ &\left. + \frac{c_1(q)z(c_1^2(q)(z+1)(z+5)+c_3(q))}{6(\omega_n^{(0)})^3} + O((\omega_n^{(0)})^{-4})\right)\end{aligned} \tag{8.17}$$

as $n\to\infty$, where by the definition

$$c_3(q) = c_1^2(q)(2c_2(q)+3) - \frac{2q}{1+q^{1/2}+q} \tag{8.18}$$

and functions $c_1(q)$ and $c_2(q)$ are given by (7.26) and (7.27), respectively.

As a result

$$\begin{aligned}\frac{1}{\kappa(\omega_n)\omega_n^z} =& \frac{(1-q^{1/2})c_1(q)}{q^{1/2}}(\omega_n^{(0)})^{1-z} \\ &\times\left(1 + \frac{c_1(q)z}{\omega_n^{(0)}} + \frac{c_1^2(q)a_1(z)}{2(\omega_n^{(0)})^2} + \frac{c_1(q)a_2(z)}{6(\omega_n^{(0)})^3} + O((\omega_n^{(0)})^{-4})\right)\end{aligned} \tag{8.19}$$

as $n\to\infty$, where

$$a_1(z) = z^2 + 2z - 2c_2(q) \tag{8.20}$$

and

$$a_2(z) = c_1^2(q)(z+1)(z^2+5z-6c_2(q)) + z\left(c_1^2(q)(2c_2(q)+3) - \frac{2q}{1+q^{1/2}+q}\right) + 9c_1^2(q) - \frac{6q}{1+q^{1/2}+q}. \tag{8.21}$$

Introducing

$$\begin{aligned} b_n = {} & \frac{1}{\kappa(\omega_n)\omega_n^z} - \frac{1}{2}\frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} q^{n(z-1)-z/4} \\ & - \frac{1}{4(1-q^{1/2})}\frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} zq^{z(n-1/4)} \\ & - \frac{q^{1/4}}{16(1-q^{1/2})^2}\frac{(q;q^2)_\infty^6}{(q^2;q^2)_\infty^6} a_1(z)q^{(z+1)(n-1/4)} \\ & - \frac{1}{24(1-q^{1/2})}\frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} a_2(z)q^{(z+2)(n-1/4)} \\ = {} & O\left(\left(\left(\omega_n^{(0)}\right)^{-z-3}\right)\right), \quad n\to\infty, \end{aligned} \tag{8.22}$$

one can see that the series

$$\sum_{n=1}^{\infty} |b_n| < \infty \tag{8.23}$$

converges absolutely and uniformly when $\operatorname{Re} z > -3$. Also

$$\sum_{n=1}^{\infty} \frac{1}{(\omega_n^{(0)})^{z-1}} = q^{(1-z)/4} \sum_{n=1}^{\infty} q^{n(z-1)} = \frac{q^{3(z-1)/4}}{1-q^{z-1}} \tag{8.24}$$

when $\operatorname{Re} z > 1$ and as a result we arrive at the following series representation for the q-zeta function:

$$\begin{aligned} \zeta_q(z) = {} & \frac{1}{2}\frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}\frac{q^{3z/4-1}}{1-q^{z-1}} + \frac{1}{4(1-q^{1/2})}\frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4}\frac{zq^{3z/4}}{1-q^z} \\ & + \frac{1}{16(1-q^{1/2})^2}\frac{(q;q^2)_\infty^6}{(q^2;q^2)_\infty^6}\frac{a_1(z)q^{3z/4+1}}{1-q^{z+1}} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{24(1-q^{1/2})} \frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} \frac{a_2(z) q^{3(z+2)/4}}{1-q^{z+2}} \\
&+ \sum_{n=1}^{\infty} \left(\frac{1}{\kappa(\omega_n)\omega_n^z} - \frac{1}{2} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} q^{n(z-1)-z/4} \right. \\
&- \frac{1}{4(1-q^{1/2})} \frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} z q^{z(n-1/4)} \\
&- \frac{q^{1/4}}{16(1-q^{1/2})^2} \frac{(q;q^2)_\infty^6}{(q^2;q^2)_\infty^6} a_1(z) q^{(z+1)(n-1/4)} \\
&\left. - \frac{1}{24(1-q^{1/2})} \frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} a_2(z) q^{(z+2)(n-1/4)} \right), \qquad (8.25)
\end{aligned}
$$

where the series defines a holomorphic function in the half-plane $\operatorname{Re} z > -3$.

We summarize our findings in the following theorem.

Theorem 9. *The q-zeta function under consideration is a meromorphic function in the half-plane* $\operatorname{Re} z > -3$. *The* $\zeta_q(z)$ *has simple poles at* $z = 1, -1, -2$ *with the residues*

$$
\operatorname*{Res}_{z=1} \zeta_q(z) = \frac{q^{-1/4}}{2 \log q^{-1}} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}, \qquad (8.26)
$$

$$
\operatorname*{Res}_{z=-1} \zeta_q(z) = -\frac{q^{1/4}}{16 \log q^{-1}} \frac{(q;q^2)_\infty^6}{(q^2;q^2)_\infty^6} \frac{2c_2(q)+1}{(1-q^{1/2})^2} \qquad (8.27)
$$

and

$$
\begin{aligned}
\operatorname*{Res}_{z=-2} \zeta_q(z) &= \frac{1}{24 \log q^{-1} (1-q^{1/2})} \frac{(q;q^2)_\infty^4}{(q^2;q^2)_\infty^4} \\
&\quad \times \left(c_1^2(q)(2c_2(q)+9) - \frac{2q}{1+q^{1/2}+q} \right), \qquad (8.28)
\end{aligned}
$$

where the $c_1(q)$ *and* $c_2(q)$ *are given by* (7.26)–(7.27). *It has no other singularities in the half-plane* $\operatorname{Re} z > -3$.

The corresponding q-Euler constant can be defined as

$$
\begin{aligned}
&\lim_{z\to 1} \left(\zeta_q(z) - \frac{1}{2} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} \frac{q^{3z/4-1}}{1-q^{z-1}} \right) \\
&\quad = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{\kappa(\omega_n)\omega_n} - \frac{1}{2} q^{-1/4} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} m \right) = \gamma_q, \qquad (8.29)
\end{aligned}
$$

which can be viewed as an analog of the classical result

$$\lim_{z\to 1}\left(\zeta(z)-\frac{1}{1-z}\right)=\lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n}-\log m\right)=\gamma. \tag{8.30}$$

From (8.25)

$$\begin{aligned}\zeta_q(0)&=\lim_{z\to 0}\,\zeta_q(z)\\&=-\frac{1}{2(1-q)}\frac{(q;q^2)^2_\infty}{(q^2;q^2)^2_\infty}+\sum_{n=1}^{\infty}\left(\frac{1}{\kappa(\omega_n)}-\frac{1}{2}\frac{(q;q^2)^2_\infty}{(q^2;q^2)^2_\infty}q^{-n}\right).\end{aligned} \tag{8.31}$$

This method can also be applied in order to investigate the analytic continuation of the similar series introduced in [24].

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Appendix A. Some asymptotics and estimates

In order to derive the asymptotic formula (2.12) one can extend the sum in (2.10) to the corresponding ${}_2\psi_2$-series,

$$\sum_{k=-\infty}^{\infty}\frac{q^{3k/2}}{(1+\omega^2q^k)^2}=\frac{1}{(1+\omega^2)^2}\,{}_2\psi_2\left(\begin{matrix}-\omega^2,-\omega^2\\-q\omega^2,-q\omega^2\end{matrix};q,q^{3/2}\right), \tag{A.1}$$

which can be transformed by the Exercise 5.20(i) of [5],

$$\begin{aligned}&{}_2\psi_2\left(\begin{matrix}-\omega^2,-\omega^2\\-q\omega^2,-q\omega^2\end{matrix};q,q^{3/2}\right)\\&\quad=\frac{(q,q,-q^{3/2}\omega^2,-q^{1/2}/\omega^2;q)_\infty}{(q^{1/2},q^{3/2},-q\omega^2,-q/\omega^2;q)_\infty}\,{}_2\psi_2\left(\begin{matrix}-\omega^2,-q^{1/2}\omega^2\\-q\omega^2,-q^{3/2}\omega^2\end{matrix};q,q\right).\end{aligned} \tag{A.2}$$

The last ${}_2\psi_2$ can be summed

$$\begin{aligned}
&{}_2\psi_2\left(\begin{matrix}-\omega^2, -q^{1/2}\omega^2\\ -q\omega^2, -q^{3/2}\omega^2\end{matrix}; q, q\right)\\
&\quad = \sum_{k=-\infty}^{\infty} \frac{(-\omega^2, -q^{1/2}\omega^2; q)_k}{(-q\omega^2, -q^{3/2}\omega^2; q)_k} q^k\\
&\quad = \sum_{k=-\infty}^{\infty} \frac{(-\omega^2; q^{1/2})_{2k}}{(-q\omega^2; q^{1/2})_{2k}} (q^{1/2})^{2k}\\
&\quad = \frac{1}{2}\left({}_1\psi_1\left(\begin{matrix}-\omega^2\\ -q\omega^2\end{matrix}; q^{1/2}, q^{1/2}\right) + {}_1\psi_1\left(\begin{matrix}-\omega^2\\ -q\omega^2\end{matrix}; q^{1/2}, -q^{1/2}\right)\right)\\
&\quad = \frac{1}{2}\frac{(q^{1/2}, q; q^{1/2})_\infty}{(-q\omega^2, -q^{1/2}/\omega^2; q^{1/2})_\infty}\\
&\qquad \times \left(\frac{(-q^{1/2}\omega^2, -1/\omega^2; q^{1/2})_\infty}{(q^{1/2}, q^{1/2}; q^{1/2})_\infty} + \frac{(q^{1/2}\omega^2, 1/\omega^2; q^{1/2})_\infty}{(-q^{1/2}, -q^{1/2}; q^{1/2})_\infty}\right) \qquad \text{(A.3)}
\end{aligned}$$

by a consequence of the Ramanujan ${}_1\psi_1$-summation formula; see, for example, [5]. The original bilateral sum in (A.1) can be rewritten as

$$\sum_{k=-\infty}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^2} = \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^2} + \frac{q^{1/2}}{\omega^4} \sum_{k=0}^{\infty} \frac{q^{k/2}}{(1+q^{1+k}/\omega^2)^2} \qquad \text{(A.4)}$$

and as a result we arrive at (2.12) from (A.1) to (A.4).

One can also derive the following formula:

$$\begin{aligned}
\kappa''(\omega) = {}& 3(q;q)_\infty^2 \; (q^{1/2}; q^{1/2})_\infty^2 \frac{(q^{1/2}\omega^2, 1/\omega^2; q^{1/2})_\infty}{(-\omega^2, -q/\omega^2; q)_\infty^2}\\
&+ \frac{(q;q)_\infty^2}{(q^{1/2};q)_\infty^2} \frac{(-q^{1/2}\omega^2, -q^{1/2}/\omega^2; q)_\infty}{\omega^2(-\omega^2, -q/\omega^2; q)_\infty}\\
&\times \left[3 - 2\frac{(q^{1/2};q)_\infty^4 (q^2;q^2)_\infty^2}{(q;q)_\infty^4 (q;q^2)_\infty^2}\left(1 + 2^4 \sum_{k=1}^{\infty} \frac{q^{3k/2}}{(1+q^k)^3}\right)\right.\\
&\left. + 2\frac{(q;q)_\infty^4}{(-q;q)_\infty^4} \frac{(\omega^2, q/\omega^2; q)_\infty^2}{(-\omega^2, -q/\omega^2; q)_\infty^2}\right]\\
&- 6\frac{q^{1/2}}{\omega^4} \sum_{k=0}^{\infty} \frac{q^{k/2}}{(1+q^{1+k}/\omega^2)^2} + 8\frac{q^{3/2}}{\omega^6} \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{1+k}/\omega^2)^3}, \qquad \text{(A.5)}
\end{aligned}$$

which determines the asymptotic behavior of the $\kappa''(\omega)$ as $|\omega| \to \infty$.

From (2.10)

$$\kappa''(\omega) = -2\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^2} + 8\omega^2 \sum_{k=0}^{\infty} \frac{q^{5k/2}}{(1+\omega^2 q^k)^3}. \qquad \text{(A.6)}$$

Rahman [19] has suggested the following transformation of this expression. Identity

$$\frac{\omega^2 q^{5k/2}}{(1+\omega^2 q^k)^3} = \frac{q^{3k/2}}{(1+\omega^2 q^k)^2} - \frac{q^{3k/2}}{(1+\omega^2 q^k)^3} \tag{A.7}$$

gives

$$\kappa''(\omega) = -3\frac{\kappa'(\omega)}{\omega} - 8\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^3} \tag{A.8}$$

and one can use Eq. (2.12) for the first term here. The second sum can be again extended to the following bilateral series:

$$\sum_{k=-\infty}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^3} = \frac{1}{(1+\omega^2)^3}\,{}_3\psi_3\left(\begin{matrix} -\omega^2, -\omega^2, -\omega^2 \\ -q\omega^2, -q\omega^2, -q\omega^2 \end{matrix}; q, q^{3/2}\right). \tag{A.9}$$

Rahman's original idea [19] is to rewrite the last ${}_3\psi_3$-series as the following special case:

$${}_8\psi_8\left(\begin{matrix} q\omega^2, -q\omega^2, -\omega^2, -\omega^2, -\omega^2, q^{1/2}\omega^2, \omega^2, -\omega^2 \\ \omega^2, -\omega^2, -q\omega^2, -q\omega^2, -q\omega^2, q^{1/2}\omega^2, q\omega^2, -q\omega^2 \end{matrix}; q, q^{3/2}\right)$$

of very-well-poised ${}_8\psi_8$-series and then to apply the transformation formula (III.38) of [5]. Here are the details. Let us consider the following transformation:

$$\begin{aligned}
&{}_8\psi_8\left(\begin{matrix} q\omega^2, -q\omega^2, -\omega^2, -\omega^2, -\omega^2, q^{1/2}\omega^2, \varepsilon\omega^2, -\varepsilon\omega^2 \\ \omega^2, -\omega^2, -q\omega^2, -q\omega^2, -q\omega^2, q^{1/2}\omega^2, q\omega^2/\varepsilon, -q\omega^2/\varepsilon \end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right) \\
&\quad = {}_5\psi_5\left(\begin{matrix} q\omega^2, -\omega^2, -\omega^2, \varepsilon\omega^2, -\varepsilon\omega^2 \\ \omega^2, -q\omega^2, -q\omega^2, q\omega^2/\varepsilon, -q\omega^2/\varepsilon \end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right) \\
&\quad = \frac{(-\varepsilon\omega^2, -\varepsilon/\omega^2, q\omega^4, q/\omega^4, q; q)_\infty}{(-q\omega^2, -q\omega^2, -q\omega^2, q^{1/2}\omega^2, -1; q)_\infty} \\
&\qquad \times \frac{(-q/\varepsilon, -q/\varepsilon, -q/\varepsilon, -q\varepsilon, -q\varepsilon, -q\varepsilon, q^{1/2}\varepsilon, q^{1/2}/\varepsilon; q)_\infty}{(-q/\omega^2, -q/\omega^2, -q/\omega^2, q^{1/2}/\omega^2, q/\varepsilon\omega^2, q\omega^2/\varepsilon, -\varepsilon^2, q\varepsilon^2; q)_\infty} \\
&\qquad \times {}_5\varphi_4\left(\begin{matrix} \varepsilon^2, q\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon^2 \\ \varepsilon, -q\varepsilon, -q\varepsilon, -q \end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right) \\
&\qquad + \frac{(\varepsilon\omega^2, \varepsilon/\omega^2, q\omega^4, q/\omega^4, q; q)_\infty}{(-q\omega^2, -q\omega^2, -q\omega^2, q^{1/2}\omega^2, -1; q)_\infty} \\
&\qquad \times \frac{(q/\varepsilon, q/\varepsilon, q/\varepsilon, q\varepsilon, q\varepsilon, q\varepsilon, -q^{1/2}\varepsilon, -q^{1/2}/\varepsilon; q)_\infty}{(-q/\omega^2, -q/\omega^2, -q/\omega^2, -q/\varepsilon\omega^2, -q\omega^2/\varepsilon, q^{1/2}/\omega^2, -\varepsilon^2, q\varepsilon^2; q)_\infty} \\
&\qquad \times {}_5\varphi_4\left(\begin{matrix} \varepsilon^2, -q\varepsilon, \varepsilon, \varepsilon, -\varepsilon^2 \\ -\varepsilon, q\varepsilon, q\varepsilon, -q \end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right)
\end{aligned} \tag{A.10}$$

and take the limit $\varepsilon \to 1$. Then

$$\lim_{\varepsilon\to 1} {}_5\varphi_4\left(\begin{matrix}\varepsilon^2, -q\varepsilon, \varepsilon, \varepsilon, -\varepsilon^2 \\ -\varepsilon, q\varepsilon, q\varepsilon, -q\end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right) = 1 \tag{A.11}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 1} {}_5\varphi_4\left(\begin{matrix}\varepsilon^2, q\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon^2 \\ \varepsilon, -q\varepsilon, -q\varepsilon, -q\end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right) \\
&\quad = \lim_{\varepsilon\to 1}\left(1 + (1+\varepsilon)\sum_{k=1}^{\infty} \frac{(q\varepsilon^2;q)_{k-1}(-\varepsilon,-\varepsilon,-\varepsilon^2,q\varepsilon;q)_k}{(q\varepsilon;q)_{k-1}(-q\varepsilon,-q\varepsilon,-q,q;q)_k}\left(\frac{q^{3/2}}{\varepsilon^2}\right)^k\right) \\
&\quad = 1 + 2\sum_{k=1}^{\infty} \frac{(-1;q)_k^3}{(-q;q)_k^3} q^{3k/2} \\
&\quad = 1 + 2^4 \sum_{k=1}^{\infty} \frac{q^{3k/2}}{(1+q^k)^3}.
\end{aligned} \tag{A.12}$$

Also

$$\sum_{k=-\infty}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^3} = \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^3} + \frac{q^{3/2}}{\omega^6}\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{1+k}/\omega^2)^3}. \tag{A.13}$$

All this together results in (A.5) after some substitutions. We leave the details to the reader.

It is worth also noting the following useful estimates:

$$|\kappa'(\omega)| < \frac{2\kappa(\omega)}{|\omega|} \tag{A.14}$$

and

$$\kappa''(\omega) < \frac{3|\kappa'(\omega)|}{|\omega|} < \frac{6\kappa(\omega)}{\omega^2}, \tag{A.15}$$

which follow directly from (2.7), (2.10) and (A.6). Indeed, by (2.7) and (2.10)

$$\frac{|\kappa'(\omega)|}{2|\omega|} = \frac{1}{\omega^2}\sum_{k=0}^{\infty} \frac{\omega^2 q^k}{1+\omega^2 q^k}\frac{q^{k/2}}{1+\omega^2 q^k} < \frac{1}{\omega^2}\sum_{k=0}^{\infty} \frac{q^{k/2}}{1+\omega^2 q^k} = \frac{\kappa(\omega)}{\omega^2}. \tag{A.16}$$

In a similar fashion in (A.6)

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{q^{5k/2}}{(1+\omega^2 q^k)^3} &= \frac{1}{\omega^2}\sum_{k=0}^{\infty} \frac{\omega^2 q^k}{1+\omega^2 q^k}\frac{q^{3k/2}}{(1+\omega^2 q^k)^2} \\
&< \frac{1}{\omega^2}\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+\omega^2 q^k)^2}
\end{aligned} \tag{A.17}$$

and

$$\kappa''(\omega) < (-2+8)\sum_{k=0}^{\infty}\frac{q^{3k/2}}{(1+\omega^2q^k)^2} = \frac{-3\kappa'(\omega)}{\omega}, \tag{A.18}$$

which results in (A.16). Also,

$$|\kappa''(\omega)| < \frac{5|\kappa'(\omega)|}{|\omega|} < \frac{10\kappa(\omega)}{\omega^2}. \tag{A.19}$$

Appendix B. Alternative forms of $c_2(q)$

Expression (7.27) for the constant $c_2(q)$ in the asymptotic formulas (7.24)–(7.25) can be transformed to a single sum in the following manner. Due to (A.13) and (III.23) of [5]

$$\begin{aligned}
&{}_5\varphi_4\left(\begin{matrix}\varepsilon^2, q\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon^2\\ \varepsilon, -q\varepsilon, -q\varepsilon, -q\end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right)\\
&\quad = {}_8\varphi_7\left(\begin{matrix}\varepsilon^2, q\varepsilon, -q\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon, -\varepsilon^2, q^{1/2}\varepsilon\\ \varepsilon, -\varepsilon, -q\varepsilon, -q\varepsilon, -q\varepsilon, -q, q^{1/2}\varepsilon\end{matrix}; q, \frac{q^{3/2}}{\varepsilon^2}\right)\\
&\quad = \frac{(q\varepsilon^2, q, -q^{3/2}/\varepsilon, -q^{3/2}/\varepsilon; q)_\infty}{(-q\varepsilon, -q\varepsilon, q^{3/2}, q^{3/2}/\varepsilon^2; q)_\infty}\\
&\qquad \times {}_8\varphi_7\left(\begin{matrix}q^{1/2}, q^{5/4}, -q^{5/4}, q/\varepsilon, -q^{1/2}, -q^{1/2}/\varepsilon, -\varepsilon, -\varepsilon\\ q^{1/4}, -q^{1/4}, q^{1/2}\varepsilon, -q, -q\varepsilon, -q^{3/2}/\varepsilon, -q^{3/2}/\varepsilon\end{matrix}; q, q\right),
\end{aligned} \tag{B.1}$$

and letting $\varepsilon \to 1^-$ one gets

$$\begin{aligned}
&1 + 2^4\sum_{k=1}^{\infty}\frac{q^{3k/2}}{(1+q^k)^3} = \frac{(1-q^{1/2})^2}{(1+q^{1/2})^2}\frac{(q;q)_\infty^4(q;q^2)_\infty^2}{(q^{1/2};q)_\infty^4(q^2;q^2)_\infty^2}\\
&\qquad \times {}_8\varphi_7\left(\begin{matrix}q^{1/2}, q^{5/4}, -q^{5/4}, q, -q^{1/2}, -q^{1/2}, -1, -1\\ q^{1/4}, -q^{1/4}, q^{1/2}, -q, -q, -q^{3/2}, -q^{3/2}\end{matrix}; q, q\right)\\
&\quad = \frac{(q;q)_\infty^4(q;q^2)_\infty^2}{(q^{1/2};q)_\infty^4(q^2;q^2)_\infty^2}\left(\frac{(q^{1/2};q^{1/2})_\infty^4}{(-q^{1/2};q^{1/2})_\infty^4} + 4q^{1/2}\frac{(1+q^{1/2}+q)(1-q^{1/2})^2}{(1+q^{1/2})^2(1+q)^2}\right.\\
&\qquad \left.\times {}_8\varphi_7\left(\begin{matrix}q^{3/2}, q^{7/4}, -q^{7/4}, q, -q, -q, -q^{1/2}, -q^{1/2}\\ q^{3/4}, -q^{3/4}, q^{3/2}, -q^{3/2}, -q^{3/2}, -q^2, -q^2\end{matrix}; q, q\right)\right)
\end{aligned} \tag{B.2}$$

by (II.25) of [5]. From (7.27) and (B.2) we obtain a single series expansion for the constant $c_2(q)$ as follows

$$c_2(q) = 4q^{1/2}(1+q^{1/2}+q)(1-q^{1/2})^2 \times \sum_{k=0}^{\infty} \frac{(1-q^{2k+3/2})q^k}{(1-q^{3/2})(1+q^{k+1/2})^2(1+q^{k+1})^2}. \tag{B.3}$$

This formula is very convenient for numerical evaluation of this constant. On the other hand, in the last line of (B.2)

$$\begin{aligned} &{}_8\varphi_7\left(\begin{matrix} q^{3/2}, q^{7/4}, -q^{7/4}, q, -q, -q, -q^{1/2}, -q^{1/2} \\ q^{3/4}, -q^{3/4}, q^{3/2}, -q^{3/2}, -q^{3/2}, -q^2, -q^2 \end{matrix}; q, q\right) \\ &\quad = \frac{(1+q)^2}{(1-q)(1-q^{3/2})} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} \, {}_4\varphi_3\left(\begin{matrix} q, -q^{1/2}, -q^{1/2}, -q^{1/2} \\ -q^{3/4}, -q^{3/2}, -q^{3/2} \end{matrix}; q, q^{3/2}\right) \end{aligned} \tag{B.4}$$

by (III.23) of [5] and, therefore,

$$c_2(q) = 4q^{1/2} \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2} \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3}. \tag{B.5}$$

By (7.10) and (7.27)

$$\alpha(q) = 3 - 2c_2(q), \tag{B.6}$$

which allows to simplify the expression for this constant in view of (B.3) and (B.5).

Numerical analysis strongly indicates that

$$\lim_{q\to 1^-} c_2(q) = 1/2. \tag{B.7}$$

For example, Gosper's Macsyma program "`namesum`" gives $c_2(0.99999) \simeq 0.49999999999616$; see also Fig. 6 for the graph of the $c_2(q)$. The proof of this result can be given in the following matter. Consider the summand in (B.5) as function of a continuous variable, say s. Then

$$\frac{d}{ds}\frac{q^{3s/2}}{(1+q^{s+1/2})^3} = \log q \;\, q^{3s/2} \frac{3(1-q^{s+1/2})}{2(1+q^{s+1/2})^4} < 0 \tag{B.8}$$

for $0<q<1$ and this function is decreasing on $[0,\infty)$. Furthermore

$$\int_0^\infty \frac{q^{3s/2}ds}{(1+q^{s+1/2})^3} = \sum_{k=0}^{\infty} \int_k^{k+1} \frac{q^{3s/2}\;ds}{(1+q^{s+1/2})^3} \tag{B.9}$$

and

$$\frac{q^{3(k+1)/2}}{(1+q^{k+3/2})^3} < \int_k^{k+1} \frac{q^{3s/2}\;ds}{(1+q^{s+1/2})^3} < \frac{q^{3k/2}}{(1+q^{k+1/2})^3}. \tag{B.10}$$

Thus

$$\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3} - \frac{1}{(1+q^{1/2})^3} < \int_0^{\infty} \frac{q^{3s/2}ds}{(1+q^{s+1/2})^3} < \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3} \tag{B.11}$$

and

$$\int_0^{\infty} \frac{q^{3s/2}\ ds}{(1+q^{s+1/2})^3} < \sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3} < \frac{1}{(1+q^{1/2})^3} + \int_0^{\infty} \frac{q^{3s/2}ds}{(1+q^{s+1/2})^3}.$$

Evaluating the integral

$$\int_0^{\infty} \frac{q^{3s/2}ds}{(1+q^{s+1/2})^3} = \frac{1}{\log q^{-1}}\left(\frac{1}{4}q^{-3/4}\arctan q^{1/4} - q^{-1/2}\frac{1-q^{1/2}}{(1+q^{1/2})^2}\right), \tag{B.12}$$

we obtain the following lower and upper bounds for the sum under consideration:

$$\begin{aligned} &\frac{1-q^{1/2}}{\log q^{-1}}\left(\frac{1}{4}q^{-3/4}\arctan\ q^{1/4} - q^{-1/2}\frac{1-q^{1/2}}{(1+q^{1/2})^2}\right) \\ &\quad < (1-q^{1/2})\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3} \\ &\quad < \frac{1-q^{1/2}}{(1+q^{1/2})^3} + \frac{1-q^{1/2}}{\log q^{-1}}\left(\frac{1}{4}q^{-3/4}\arctan q^{1/4} - q^{-1/2}\frac{1-q^{1/2}}{(1+q^{1/2})^2}\right). \end{aligned} \tag{B.13}$$

By the Squeeze Theorem

$$\begin{aligned} &\lim_{q\to 1^-}(1-q^{1/2})\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3} \\ &\quad = \lim_{q\to 1^-}\frac{1-q^{1/2}}{\log q^{-1}}\left(\frac{1}{4}q^{-3/4}\arctan q^{1/4} - q^{-1/2}\frac{1-q^{1/2}}{(1+q^{1/2})^2}\right) = \frac{\pi}{32}. \end{aligned} \tag{B.14}$$

From (B.5) and (1.5)

$$\frac{c_2(q)}{c_1(q)} = 8q^{1/4}(1-q^{1/2})\sum_{k=0}^{\infty} \frac{q^{3k/2}}{(1+q^{k+1/2})^3}, \tag{B.15}$$

which implies (B.7) in view of (B.14) and (1.6).

The graph of the $c_2(q)$ in Fig. 6 indicates that $\lim_{q\to 1^-} c_2'(q) = 0$. The author was unable to give a rigorous proof of this result yet.

Conjecture 2. $\lim_{q\to 1^-} c_2'(q) = 0$.

Appendix C. Monotonicity of $c_1(q)$

The graph of the $c_1(q)$ in Fig. 6 reveals the monotonicity of this function on $[0, q]$. Here we give a rigorous proof of this result. Differentiating (1.5) one gets

$$\frac{dc_1(q)}{dq} = -2c_1(q)\left(\frac{d}{dq}\log\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - \frac{(1+q^{1/2})^2}{8q(1-q)}\right), \tag{C.1}$$

where

$$\begin{aligned} \frac{d}{dq}\log\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} &= \sum_{k=0}^{\infty}\frac{(2k+1)q^{2k}}{1-q^{2k+1}} - \sum_{k=0}^{\infty}\frac{(2k+2)q^{2k+1}}{1-q^{2k+2}} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)q^n}{1-q^{n+1}}. \end{aligned} \tag{C.2}$$

The last sum can be transformed as follows:

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)q^n}{1-q^{n+1}} &= \sum_{k=0}^{\infty} q^k \sum_{n=0}^{\infty}(-1)^n(n+1)(q^{k+1})^n \\ &= \sum_{k=0}^{\infty}\frac{q^k}{(1+q^{k+1})^2} \end{aligned} \tag{C.3}$$

in view of the geometric series and its derivative

$$\sum_{n=0}^{\infty}(-1)^n(n+1)z^n = \frac{1}{(1+z)^2} \tag{C.4}$$

with $z = q^{k+1} < 1$. Thus

$$\frac{d}{dq}\log\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{k=0}^{\infty}\frac{q^k}{(1+q^{k+1})^2} \tag{C.5}$$

and the substitution of (C.5) into (C.1) results in

$$\frac{dc_1(q)}{dq} = -2c_1(q)\left(\sum_{k=0}^{\infty}\frac{q^k}{(1+q^{k+1})^2} - \frac{(1+q^{1/2})^2}{8q(1-q)}\right). \tag{C.6}$$

In order to prove the monotonicity of the $c_1(q)$ we need to show that

$$\sum_{k=0}^{\infty}\frac{q^k}{(1+q^{k+1})^2} < \frac{(1+q^{1/2})^2}{8q(1-q)} \tag{C.7}$$

for $0 < q < 1$.

The last inequality can be proven in the following manner. One gets

$$\frac{q^k}{(1+q^{k+1})^2} < \frac{1}{q}\frac{q^{k+1}}{(1+q^{k+1})(1+q^{k+2})} = \frac{1}{q(1-q)}\Delta\left(\frac{1}{1+q^{k+1}}\right), \tag{C.8}$$

where by the definition $\Delta l(k) = l(k+1) - l(k)$. Therefore

$$\sum_{k=0}^{\infty} \frac{q^k}{(1+q^{k+1})^2} < \frac{1}{q(1-q)} \sum_{k=0}^{\infty} \Delta\left(\frac{1}{1+q^{k+1}}\right) = \frac{1}{1-q^2}. \tag{C.9}$$

The final step is to show that, in fact,

$$\sum_{k=0}^{\infty} \frac{q^k}{(1+q^{k+1})^2} < \frac{1}{1-q^2} \leqslant \frac{(1+q^{1/2})^2}{8q(1-q)} \tag{C.10}$$

on $[0,1]$. The last inequality is equivalent to

$$(1-q^{1/2})^2(1+4q^{1/2}+q) \geqslant 0, \tag{C.11}$$

which holds for all $0<q\leqslant 1$. As a result

$$\frac{dc_1(q)}{dq} > 0, \quad 0<q\leqslant 1, \tag{C.12}$$

which completes our proof of the monotonicity of the $c_1(q)$.

In a similar fashion

$$\frac{q^k}{(1+q^{k+1})^2} > \frac{q^k}{(1+q^k)(1+q^{k+1})} = \frac{1}{(1-q)}\Delta\left(\frac{1}{1+q^k}\right), \tag{C.13}$$

$$\sum_{k=0}^{\infty} \frac{q^k}{(1+q^{k+1})^2} > \frac{1}{2(1-q)}, \tag{C.14}$$

and in view of (C.9) and (C.14)

$$\frac{1}{2(1-q)} < \sum_{k=0}^{\infty} \frac{q^k}{(1+q^{k+1})^2} < \frac{1}{1-q^2}. \tag{C.15}$$

By the Squeeze Theorem then

$$\lim_{q\to 1^-} (1-q) \sum_{k=0}^{\infty} \frac{q^k}{(1+q^{k+1})^2} = \frac{1}{2}. \tag{C.16}$$

Numerical analysis strongly indicates that $\lim_{q\to 1^-} c_1'(q) = 1/2\pi$. For example, Gosper's Macsyma program "`namesum`" gives $c_1'(0.99999) \simeq 0.50000641104998/\pi$. The author was unable to give a rigorous proof of this result.

Conjecture 3. $\lim_{q\to 1^-} c_1'(q) = 1/2\pi$.

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